THE STUDY OF INTUITION AS AN OBJECTIFYING ACT FROM A HUSSERLIAN PERSPECTIVE, IN THE CASES OF THREE PROSPECTIVE TEACHERS OF MATHEMATICS

ANDONIS ZAGORIANAKOS

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Abstract

The aim of the research is to rethink how students of mathematics learn, by employing the ‘late’ Husserl’s theory of knowledge. To do this the mathematical investigations of three prospective teachers of mathematics are explored, focusing on ‘objectification’ and ‘intuition’, thinking about how students acquire new knowledge in order to resolve mathematical tasks; in particular how students come to grasp mathematical objects in order to conceptualise the tasks. For this purpose I employed Husserl’s phenomenological attitude and his methods of reduction and bracketing, informed by Merleau-Ponty’s extension of the Husserlian frame. The course from where the data were collected included thirteen students, and the teacher facilitated my phenomenological perspective, by deliberately suspending guidance and any other intervention apart from introducing the operational aspect of the tasks.

By using the Husserlian theory and methodology and by following the teacher’s non-intervention strategy I managed to track the ‘moments’ of objectification and the critical role of intuitions—in the Husserlian sense—in the process of objectification. The embodied, founding powers of the living body (the body-subject) and the pre-reflective and reflective intentional forces manifested their significance for the students’ objectification processes. Most importantly, intuitions appeared as the critical acts that enabled each objectification to take place.

In summary, the main findings of the research and the related contributions of the study to knowledge are the following:

Intuitions are critical objectifying acts, preparing as well as constituting mathematical objects.

Three genetic features of intuitions are identified, thus allowing their tracing as such.

Empirical and abstract intuitions were shown as essentially interrelated, and the description of the transition from empirical to abstract knowledge through according objects was exemplified in a number of cases.

The general structure of the (Husserlian) abstract intuitions was clarified, leading to suggestions for teachers to introduce abstract objects in accordance to the aforementioned structure.

A novel phenomenological gaze on the mathematical learning experience is introduced, one that transforms ready-made

mathematical objects to objectified lived experience. The contribution to knowledge suggested by this gaze is that it takes into account the complexity and richness of the learners' lived worlds, that it has the potential to reorient teaching practice into a student-oriented inclusive praxis, and finally, to enable the researcher to cash in Husserl's theoretical and methodological reflections as a "working philosophy".

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In order to arrive at what you do not know
You must go by a way which is the way of ignorance.
In order to possess what you do not possess
You must go by the way of dispossession.
In order to arrive at where you are not
You must go through the way in which you are not.
And what you do not know is the only thing you know
And what you own is what you do not own
And where you are is where you are not.

T. S. Eliot – "East Coker"

Preface

This research is the result of an exploration, of a parallel study of theoretical—phenomenological—concepts that Husserl's theory provides, in relation to what Husserl calls the empirical 'self-evidence'; namely focusing upon observations and the accounts of the thinking strategies that the work of three prospective teachers of mathematics have provided, thus bringing evidence that the theory availed.

The evidence brought about and analysed here is made possible due to the illuminations afforded by the chosen theoretical and methodological filters, which were realised due to the phenomenological attitude that I employ. The Husserlian theory and methodology that I principally draw on is largely unknown to our field of mathematics education research, and thus my endeavour here is also linked to an introduction to theoretical and methodological ideas that could be useful for further study and analytical praxis. In other words, through the text that follows I aim at explicating a novel phenomenological gaze on the mathematical learning experience, as a first novelty of this study, prior to the findings that the three case studies will accordingly present and explicate.

The *second* novelty is brought by the main theme of my study, namely the exploration of the essential part that *intuition* plays in the *objectification* process—

the commencing of objectification in particular—which originates from the students' lived learning experiences. The novelty is the identification of *intuition* as a *critical objectifying act,* due to the Husserlian theoretical and methodological framework that I adopt. The three students' lived learning experiences are brought to the surface as "quivering fish" (Merleau-Ponty, 2002a, p. 14)—i.e. alive—by means of the aforementioned theoretical perspective and methodology; for this reason the detailed analysis of the learning episodes is framed by a necessarily extended theoretical presentation, in order to introduce the necessary concepts for the better grasping of the analysis.

Finally, since Husserl's books are notoriously lacking in examples (Moran, 2010, p. 63), my study aspires to serve also as an *exemplification* of subtle, constitutional notions of mature Husserl's theory of perception, in particular as it was radically extended through Merleau-Ponty and his primacy of perception.

Introduction

This introduction is about a research study that is focused on *how the learner learns,* attempting to delve as deeply as possible into this phenomenon as does the infamous Meno's paradox,[1] which concerns the *possibility* of knowledge. The theoretical ground of my research is a tribute to Husserl's strenuous, unresting reflections, to the philosophically undertaken unresolved tasks that he embarked on and the paths that he inaugurated. The empirical aspect of the research is shaped as the study of three cases, three prospective teachers of mathematics, analysed using the same theoretical lens and the according methodology.

Three learning episodes are at the focal point of the analysis, emerging as distinct and complementary cases of how learning takes place. The focus will be on the devising of objects (*objectifications*) on behalf of the learners (prospective teachers of mathematics)—in order to tackle their tasks—and the corresponding

[1] Meno's paradox concerns the possibility for the acquisition of new knowledge. See sections 1.2.2 (for the introduction and the theoretical consideration of the paradox), 3.1.1, 3.2, 4.3.2 for the discussion of the solution that Husserl gives to Meno's paradox, and the evidence that the three cases give in support of Husserl's theory of perception.

intuitions that brought them to surface, from my Husserlian perspective, and after the appropriate *reduction* and *bracketing,* as they will be clarified in the following chapters. But a few words before I'll technically unfold these three cases are needed, concerning the general aims that support the rationale of this study.

Anyone engaged more or less in mathematical enquiry knows that the results of this endeavour are usually presented as stress free, as an almost 'natural' process. In the textbooks, mathematics is ordered in a linear way and presented as a unity in the sense of a ready-made product (Steinbring, 1991). Teachers present theorems to the students unaware of the etymology of the word, which is associated with the final view (thea) at the top of a hill or a mountain, after the laborious climbing has taken place. A similar phenomenon takes place when teachers encounter their students' thoughts and suggestions, since the teachers gauge them according to their perception of mathematics and mathematical knowledge, neglecting the particular paths that their students take and their *key ideas.* The qualification of these key ideas in order to become visible is the target core of my present research. In the analysis of the data we will shed light on the *figuration process* of the learner's world-as-lived in relation to mathematical knowledge; and trace the origins of learning in the reactivation of cultural objects (see the parabola in Part 3, Ch. 4), first signal the immediate access of the individual to her lived world, from which the learner acquires the texture and the material of her thoughts. Using a corresponding phenomenological data analysis my current research has a four-fold aim:

- To accentuate the *given* aspect of learning mathematics as a non-prepackaged experience of the learner's world-as-lived.
- To manifest how intuition may be realised as a critical objectifying act, operating as the link between formal mathematical knowledge and the learner's lived-experience.
- To explore and to exemplify the role of different kinds of intuitive acts in the first stages of objectification. To attempt an inquiry into the 'things themselves', the learner's lived-experience that becomes the material for empirical and abstract (mathematical) objects, and to

explore essential actual and structural relations between these objects.

- To suggest possible implications of a Husserlian phenomenological perspective for teaching practice.

From the aforementioned set of interrelated aims the three cases of the prospective teachers of mathematics that will be analysed can be technically (operatively) summarised as follows, as bringing different and similar evidence on the objectification – intuition link that is the theme of the study:

- An apodictic[2] essential intuition and a corresponding objectification from a gifted student, who fully enjoyed the sessions. The intuition unlocked the task for the student and was responsible for the plethora of mathematical objects that followed.
- An empirical intuition, followed by an abstract one that retained the initial theme, while producing more advanced mathematical objects. The intuitions and the legitimised results were able to shake the student's perception of learning and teaching, as well as transform the student's frustration to a full appreciation of the course.[3]
- And a 'stream' of intuitions (and according objectifications), supported and primarily grasped through embodied re-presentations of the student's lived world, through a diagram that was realised due to a pivotal sensuous intuition.

Research questions – preliminary delineation of mathematical objects

My investigation started from the desire to investigate the acts and thoughts of learners of mathematics in order to tackle mathematical tasks, in terms of the students' own decisions how to develop their own understandings. The different

[2] Pertaining to proof; clearly established or of clear demonstration.

[3] The teacher's role was critical here, since probably nothing would have been achieved without his legitimisation of her results.

linguistic and cultural environment and my participant-observer role gave me a good chance to put out of play my teaching perceptions, and focus on (learning) features that would pass unnoticed in my familiar Greek environment of teaching-mathematics. I and the teacher already believed in the diversity of understandings (and of the expressions of understandings) of the learners of mathematics, and we were wishing to trigger them due to our (the teacher's and mine) non-guiding teaching and participation strategies respectively.

Since the focus of the study will be on the devising of objects (*objectifications*) on behalf of the learners a preliminary description of the perception of mathematical objects in this study is necessary. So from the perspective of my study mathematical objects are anything intended for and resonant with formal mathematical reasoning that stands out for the *embodied* mind 'for a reason', namely due to intentions that are ~~(~~thus~~)~~ fulfilled. The mathematical objects are prepared in pre-reflective consciousness before they appear as posited objects; and they are either reactivating previously established concepts in the mathematical canon, or instigating new paths, towards new scientific constitutions (and eventually becoming part of the body of tradition). Mathematical objects that already belong to the body of mathematical tradition transcend the learner's understanding, yet they are constantly in motion, precisely due to learning and innovative experiences that I just indicated. Therefore, there is no sense in analysing them from a straightforward, natural attitude, and I would overlook critical features of their emergence if I would employ positivistic research methods, since these methods would detach them from the life-world that brought them to life in the first place. Nor could I rely on sociocultural approaches that neglect the role of subjectivity, because they perceive the latter as an *a priori* idealistic philosophical gesture.[4]

[4] Yet there is a dialectic relation between the phenomenological approach suggested here and the sociocultural approach and in particular activity theory (Shvarts & Zagorianakos, 2015, *Crossroads of Phenomenology and Activity Theory in the study of the number line perception*, CERME9 presentation) that is not opened in this study, due to space limitation and the need of further research results (it is research study in progress).

Another aspect that determined some of my research questions was the need for an appropriate theory and an accordingly adequate methodology that could address their complexity from the perspective of the learner. So my *research questions*—which arose as soon as I started my research—concerned how students learn from the viewpoint of the mathematical objects that they devise in order to tackle mathematical tasks. My *research questions* are as follows:

- How do the students devise mathematical objects in order to tackle mathematical tasks?
- What were the students' practices as expressions of their lived experience throughout the learning situations, and in particular during the moments of tension, when new mathematical objects emerged for the students? This exploration involved parenthesising/minimising the teacher's intervention and my participation, in order to lay bare the students' own practices. For the same reason the first-person perspective was chosen.

The aforementioned student oriented research questions employed implicit philosophical and methodological questions, in asking questions about student learning, which were the following:

- How is new knowledge possible for the students,[5] given that they do not know what they are looking for? In other words I explore how the students come to new knowledge (for them) and how they know that what they found is what *they* were looking for.
- Are there ways of analysing mathematical objects as phenomena of lived reality, and not as ready-made objects of our world, 'already existing' and 'ready to be transferred'? In philosophical terms, is there something in the framework that I will use that allows me, as a mathematics education researcher, to think differently about the issues of learning mathematics, in particular of acquiring abstract knowledge through tasks?

[5] This research question is related to Meno's paradox, which was introduced in a previous section and which will be unfolded in the following sections.

- Can theoretical and methodological frames that claim to give thorough understanding of the learning experience, be put to the test of actual learning praxis and give results? That is results that concern the texture and the structure of this elemental cognitive 'moment' of objectification. Such results might exemplify an adequate theoretical/methodological paradigm that could be used effectively for the open exploration of the students' learning situations, with respect to their personal development as part of the development of mathematics.
- What are the implications for bodies of knowledge, pedagogic practice, curriculum development, and the approach of the 'classroom' as a complex intentional work unit, i.e., questioning what constitutes a learning space.

While I was focused on moments when objects were devised by the learners and allowed or even pushed their mathematical explorations further, I was led to explore the students' key ideas and their origination. And intuitions were found to be constant companions to the origination of these key ideas and the mathematical objects that followed them; hence, I was driven to the investigation of intuitions, in relation to objectification. Thus another research question completed the previously mentioned ones, as intuition became prominent in my research:

- What is the part that intuitions play in the students' objectification process, and in its inaugural steps in particular?

The research questions—either focusing on the students or those involving related philosophical issues—concern the *how* of the emergence of mathematical objects; and it will be studied by putting out of play the direct (traditional) approach to teaching, and by adopting the first-person perspective. This parenthesising of teaching guidance was a well-known practice for the teacher and it facilitated tremendously the theoretical (phenomenological) frame of my research. From my research point of view, this suspension of traditional teaching practices is also performed in order to serve the teachers' own practices, by suggesting a new perception of their learners' practices, after the investigated phenomenon (intuitions

as objectifying acts) will be fully unfolded. The next section will introduce important terminology appearing in the research…

Methodological considerations – Introduction of intuition in the Husserlian sense

The italicised words in this section concern terms that will be defined in the coming sections. Continuing on the technical (operative) level, before entering the world of intuition, objectification and their close links, I will now give a preliminary description of my methodological approach and the inevitability of becoming involved again and again in paradoxes, such as Meno's paradox, which will be unfolded in the following sections and concerns the possibility of knowledge. I will use Husserl's words, since my research is on the path of the phenomenological-transcendental radicalism that Husserl talks about in the following extract. The paradox that Husserl exposes is how a philosophy that denies the possibility of any ground of things that can be *taken for granted,* does in fact acquire ground for itself:

> [W]hat is peculiarly proper to the essence of the incipient philosophy of this phenomenological-transcendental radicalism is that, as we have said before, rather than having a ground of things taken for granted and ready in advance, as does objective philosophy, it excludes in principle a ground of this or any other sort. Thus it must begin without any underlying ground. But immediately it achieves the possibility of creating a ground for itself through its own powers, namely, in mastering, through original self-reflection, the naive world as transformed into a phenomenon or rather a universe of phenomena. Its beginning course, like that carried out above in rough outlines, is necessarily one of experiencing and thinking in naive self-evidence. It possesses no formed logic and methodology in advance and can achieve its method and even the genuine sense of its accomplishments only through ever renewed self-reflections. Its fate (understood subsequently, to be sure, as an essentially necessary one) is to become involved again and again in paradoxes, which, arising out of uninvestigated and even unnoticed

> horizons, remain functional and announce themselves as incomprehensibilities. (Husserl, 1970a, p. 181)

The paradox that Husserl introduces is resolved by the transformation of the naïve world into a phenomenon "or rather a universe of phenomena". My difficulty in approaching the phenomena of the students' actions was resolved by employing phenomenology as a *methodology* (cf. Carr, 1999, p. 97, Zahavi, , pp. 667-68),[6] by suspending our beliefs in the objects of knowledge as existing ("as taken for granted unquestioningly"),[7] as ready-made artefacts ready to be transferred; and by focusing on their appearance, while studying the objects' emergence in the subjective intentional realm that constitutes them as such, ready to be communicated with other objects, by "ascrib[ing] to them, as to other objects, actual (veritable) being" (Husserl, 1983a, p. 41).

A realm that is not a closed box, but transcendent "through and through", as Merleau-Ponty contends, radically extending Husserlian concepts, such as the intentional force that operates pre-reflectively (*operative intentionality*). This intentional force synthesises what will be(come) objectified, like the symmetry of a drawing, felt by the body as such and turning its attention towards the drawing where the symmetry is detected (and thus *thematising* it, as Husserl would call it). Merleau-Ponty also accentuates the embodied understanding that precedes objectification (i.e. pre-objective) and the logical categories (pre-logical), and it renews cultural significations. Cognition for Merleau-Ponty and Husserl is seen as grounded on the embodied contact with one's lived-world through the body-subject (the *living body* or *lived body*), which organises my awareness, in particular the passive one, like my awareness that not only am I writing these lines in a room, but that this room is on a floor of a building. Even before my attention is directed to the latter, it is given to me, for a choice (judgment) that I may have to consider; in

[6] Concrete arguments for Husserl, Heidegger and Merleau-Ponty's common perception on the phenomenological methodological instruments of reduction and bracketing can be found in Zahavi, 2008.
[7] Husserl, 1970a, p. 74.

contrast with the *body-object* (Körper), which is a theme of the so called natural sciences.

The Husserlian/Merleau-Pontyian approach to conscious cognitive acts that was just delineated takes into account that postulate one for the phenomenological perspective is that consciousness is *intentional,* it is always consciousness *of*, aiming at objects. The objects that consciousness aims at, fulfil a complex web of intentions, which bring them to surface, but "it is the non-reflective consciousness which renders the reflection possible; there is a pre-reflective cogito which is the condition of the Cartesian cogito" (Sartre, 2010, p. 9). It is in this sense that objectification is a breathing process for consciousness, which constitutes endlessly new artefacts, according to new tasks and tailored to the learner's manners, her objectification history in a particular domain like the mathematics.

From this perspective the intuitions' critical part in the commencing of objectification makes a strong link towards many directions, like the students' lived worlds, previous conceptions and misconceptions, re-cognition, renewal of concept understanding (reactivation) etc.. My research aims at disclosing the richness of the emergence of objectification through the revealing of the essential part that intuition in the Husserlian sense plays in it. Examples will be given and different sorts of intuitions will be analysed more than once, in the same or different cases; particular intuitions/objectifications will be in focus, namely the ones leading to abstract, mathematical objects. And such objects are traced in the learners' lived experiences, emerging from empirical, idiosyncratically employed and taken into action materials, from profiles of the real, as-lived by the learners themselves, from the first-person perspective.

In the following chapters I engage in "experiencing and thinking in naive self-evidence" (op. cit.), moving through a process that could be read as renewed reflections on the learning episodes, while operating with a method that progressively *puts out of play* everything but the object and the objectifying act. The relation between the intending act and the intended object, as I approach it in my study, drawing mainly on Husserl, could be prefigured as follows:

Conscious acts are characterised by their *directedness* to objects, a phenomenon called *intentionality* [of consciousness]. When the learner constitutes an object, as she realises through conscious acts that it *fulfils* the intentionality that was directed to it,[8] and when she brings it to existence with *immediacy*, i.e. with a certain sense of *instantaneity* (or *directness)* and *non-mediation,*[9] we have the case of intuition in the Husserlian sense of the term, which I adopt here and will be explored throughout this study. What is constituted in the aforementioned process appears as objective and it is ready to be circulated in the learner's community, starting from her peers and her teacher, who is also responsible for the legitimisation of the constituted objectivity that comes with the learner's object. This fundamental interplay, which starts in a mathematics classroom, at home or anywhere else, and which generates mathematical objects in the learner's attempt to conceptualise mathematical phenomena, is at the centre of my study—i.e. the phenomenon of object constitution in the Husserlian sense—in my attempt to cast light on the decisive role that intuitions play for the objectification process.[10]

Since the aim is to take nothing for granted in my research, the teacher's masterfully performed withdrawal from instruction, and the open-endedness of his mathematical tasks helped me immensely to bracket the teacher's part in order to focus on the students' own ideas, strategies and judgments. But I am also especially indebted to the teacher, for

- giving me full license to participate and record the sessions;
- giving me full access to the students' coursework;

[8] The object in this Husserlian view becomes an object when the subject realises that it fulfils the intentionality that was striving for fulfilment.

[9] I am using *immediacy* in the double sense of the word, i.e. *with instantaneity* and *without intervention*.

[10] The italicised words mentioned in the paragraphs after Husserl's long quotation are also key terms for the theory and the analysis of the learning episodes; but they should be understood quite literally now, without feeling that any 'essential meaning' is missed. Husserl himself used *horizons, intentions* and *intentionality* (roughly as a web of intentions) in much an every-day sense. And the reader is expected to gradually acquire the specificities of the meanings involved in the terms mentioned here according to the context that they will appear throughout the text, in theoretical, methodological and analytical adumbrations that will be presented in the following parts of the study.

- introducing me nicely to the students, as a Greek teacher of mathematics who does postgraduate research, trying to understand the students' way of thinking.

My research is the result of several reflections, after revisiting the theory, the data sources, the data and the findings, the theory and the findings etc. (cf. Graph 4 in Part 2, §2.2). The aforementioned movement was shaped by my methodological choices and the methods employed; but the methodology was not an imposed, one-dimensional instrument that was preconceived as appropriate; rather, the research achieved "its method and even the genuine sense of its accomplishments only through ever renewed self-reflections" (op. cit.). In other words, the reflections on previously analysed material brought new findings on the surface, and enriched both the data as a whole and the range of the methods that were implicated; hence the methodology and the data analysis were critically interrelated, as the latter was forwarded by the former. And it is not excessive to say that the self-reflections concerning the findings have been continuous, up to the submission of the dissertation; in other words, I see the thesis itself as a 'moment' within broader research, while already sufficiently matured in order to present original and interesting results for mathematics education research.

PART 1 – Theoretical framework

A short introduction and summary of the five chapters of the study

The division of the theory section of the study into five parts is deemed useful for the following reasons:

- The extensive (given the space limitations) introduction of Husserl's theory of perception, which is necessary in order to grasp the particularities of the data analysis.
- The Husserlian approach to objectification and intuitions 'separately', although separabillity is not firmly achievable, since it is intuitions as objectifying acts that is the theme of exploration here.
- The Husserlian approach as a critical alternative to functionalistic approaches towards cognition that abundant in mathematics education, residing in behaviouristic approaches in their blindness to the phenomenological apparatus of intentionality.[11]

The first chapter will introduce necessary concepts and terminology that will be unfolded in the following sections, it will indicate the Husserlian concepts that were of particular interest for Merleau-Ponty (and which are relevant to this study), and it will delineate—using the aforementioned concepts and terms—a Husserlian and Merleau-Pontyian response to the Cartesian doubt, as an outline of the theoretical and exemplary explication of the distinction that will follow. In other words it is, from the genetic point of view, the starting point of the aporia that drives this study. The aporia that led me to the exploration of the ground of learning. The question was "how do we manipulate our lived experience, how do we arrive at new knowledge (for us)?" "How does abstract knowledge come about? "How is the learner's lived experience involved in what we see as final ready-made result?" "How

[11] For further explication of my phenomenological approach see the three reasons that legitimise a theory of transcendental descriptions (one exculpatory, one contemplative and one polemical) in Sokolowski (1983, pp. 221-222).

does *speaking speech* emerge and what resources does it use, and finally, how Husserl's theory may give access to critical areas of cognition that are ignored or misunderstood by Kantian and Cartesian approaches, still spread in mathematics education research; having in mind a constructive dialogue, rather than a confrontation.

The second chapter will clarify the distinction between Husserlian and Kantian/constructivist frames of experience, and elaborate the Husserlian perception of experience–as–lived, radically interpreted by Merleau-Ponty and his focusing on the operative powers of the lived body. The Husserlian approach with the Merleau-Pontyian primacy of perception, which is adopted in my study, is contrasted with the already categorically afforded Kantian perception of experience, which demonstrates and materialises the presents of lived-reality separately from sensibility, in an allegedly complete complementarity. Some clarifications towards contemporary materialist and empiricist studies that use phenomenological frames will be given, and finally, the Husserlian *principle of all principles* will be introduced.

The third chapter introduces objectification and *Meno's paradox.* A literature example is used in order to introduce Husserl's intentionality, the process of objectification as based on founded and founding acts, and the exemplary application of the phenomenological reduction, which will be employed in the analysis of the data.

The forth chapter also introduces the milieu of objectification and intuition on the ground of the life-world of the learner. It manifests the critical part of the just-have-been (*retention*) and (particularly) the just-about-to-arrive (*protention*) moments of the present 'moment', in the confirmation or rejection, and the anticipation respectively, of what continually enters the field of perception. It is shown how the theory acknowledges and allows the potential for new thematisations that retention and protention hold for the conscious present—not in a fully-fledged manner, yet in direct contact with the lived world of the learner and announcing the .

The concluding fifth chapter starts by summarising objectification and key concepts for this study, like the living body. It further demonstrates the distinction between the Husserlian and the Kantian approaches, using terminology that has now been explicated. Finally, the Husserlian approach to *intuitions* that is adopted in this study is presented, the three *genetic* features/properties of intuitions that allow them to be detected as such, and the different kinds of the Husserlian intuitions that will be studied. The chapter concludes with a summarising Table 1 and a Graph 3, which will be poles of reference throughout the study.

CHAPTER 1. Introduction to the theory, the aims and the terminology of the study

1.1 Further clarifications necessary for the better grasping of the concepts and the terminology that will be used throughout the thesis

Before starting introducing the theory and the aims of the study, further clarification of terms that will be used is necessary, due to the strong theoretical aspect of this research. The unfolding of the terms will take place in the context of the introduction to the theory, which is related to my research on Husserlian intuitions and their critical role in the objectification process. The terms will be italicised and particularly indicated (i.e. put in brackets, after a short description of their meaning), but not expected to be understood in their full sense, which will be later unfolded. As Husserl (1983a, p. 201) put it,

> [I]n phenomenology, at the beginning, all concepts or terms must remain in flux in a certain way, always at the point of being differentiated in accord with the progress of the analysis of consciousness and the cognition of new phenomenological strata within what is at first seen in undifferentiated unity.

Therefore, our common sense understanding of the terms will suffice for the moment, and it will be polished as we will further expand our reading, and after the decisive analytical employment of the terms, which will clarify the theory that would otherwise be fruitless and empty according to Kant's ever useful claim.[12] The three learning episodes where the terminology will be brought to life (Part 3) aim at becoming the culmination of the theory, and their analysis and discussion (Parts 3 and 4 respectively) an exemplary path for research and teaching practice. As it was in the Platonic dialogues, where a principle message is carried out in the dialogical/dialectic form of inquiry, deeply rooted in the texts themselves, our expected gain as readers starts from the saturation of the theory and the methods

[12] "Thoughts without content are empty, intuitions without concepts are blind" (Kant, 1998, B75, pp. 193-194).

used. The findings of the data, approached as they were by the particular theory and methodology bring about, in my view, the two-folded novelty of the findings themselves and of the theory/method that enables their approach. The research questions started in a naïve way.

Having said at the start of this section that the focus of the study is the objectification process and the intuitions that are involved in it, and having acknowledged my theoretical debt to the Husserlian reflections and acquisitions in perception and the possibility of abstract knowledge, I need to clarify, already from the beginning that my theoretical 'devotion' is not an unreflective or uncritical one. Husserl's approach to perception was, at least seemingly, too tight on the *ego pole* (the constituting subject) and the *object poles* (the constituted objects), an approach considered by many as a dualistic one (e.g. Seamon, 2010, p. 2,).

But Husserl thematised and Merleau-Ponty explored the synthesising[13] potential of the tension between the subject and the object poles (Husserl, 1970a, 1973, 2001; Merleau-Ponty, 2002a, b, 1964a). Husserl related these two poles by the subjective (noetic) experience (1970a, p. 171), in which relation no one aspect is prior to or apart from the other – hence not so much a mediation as between two external things, as a co-existence where each arises one with the other in the act of being directed towards. This is the relation between the thinking act and the thought object, between the *intention* and the *intended,* already starting before any thought emerges, as the embodied consciousness operates (before the subject/object rift).[14] I think of the little red ball that I suspect lying under the wooden table and my body coordinates to give me the appropriate view; I bend, I turn my head, I move my eyes towards certain 'corners', informed by recollection— or *intuitive* recollection, as Husserl (2001) calls it, since it is fulfilled by previous

[13] This tension also demarcates the field for an inherent (*immanent* in Husserl's terms) dialectics of lived-experience, in the context of the lived-world. It is where the ground starts becoming groundless, as it yields authorship and authorisation to the *allure* of the objects (Husserl, 2001, p. 196). It is what Husserl calls affection (ibid.), which progresses to a "striving toward self-giving intuition, disclosing more and more of the self of the object, thus, striving toward an acquisition of knowledge, toward a more precise view of the object" (ibid.).

[14] Cf. Zahavi, 2008, p. 670.

direct contact with this small red ball, under this very table, which is intentionally recalled for this purpose.

In all this my living body operates as an organ of perception, unseparated by what will eventually emerge as thought ("it could be behind this bamboo basket, as it was the other time"). And by putting aside this *orchestrated* operation—cf. the students' diagrams in DeFreitas and Sinclair, 2012 for another example of this exploration on lived space[15] that my study also argues for—which is at the same moment the condition for any sort of epistemic knowledge to be constituted, anyone whatsoever, mathematics and even pure, formal logic included, according to Husserl (1970a, p. 141 –full citation follows in Part 1, §1.2.2),[16] we do not pose our cognitive enquiries in a full sense, far from it. By ignoring the origins of our understandings from the view of the 'I' pole (the subject of learning), due to an equivocal suspicion towards subjectivism, we do not get wiser about how the learner learns. If we allow only the products of reflection to enter our cognitive, mathematical scope we miss a critical foundational aspect of how subjectivity perceives the world, while *in* and *for* the world, how each one of us enters the adventure of *new* knowledge, first of all new-for-us, as we are *in* and *for* the world. This transition from the pre-reflective to the reflective 'materials' of the learners' cognitive efforts are in focus when I analyse how the three students deployed their strategies, and it will be the main background of this analytical exemplification of *intuition* in the Husserlian sense, as it operates and acts in processes of *objectification,* in its inaugural stages. And the opening moment of this experience is the immediate perception of a profile of the real, the intuitive *primal impression*.

Husserl makes clear that what we call present is not only the aforementioned immediate perception of the real, that the lived experience of the present is not structured simply as a succession of these immediate perceptions (termed as p*rimal*

[15] It would probably be interesting to compare and contrast these diagrams with Mary's diagram in Part 3, Ch.4.

[16] Also, see Part 1, §1.3.2 for an extensive analytical exemplification, drawn from the literature and setting the theoretical/methodological ground for the data analysis.

impressions by Husserl). The present is *already* a multiplicity that co-exists,[17] *co-presented* in the case of the primal impression and *co-presentified* (in Husserlian terminology) in the other two cases: since for each primal impression there is simultaneously a *just-having-been* of a primal impression—which is a presentification[18] that is temporalised accordingly and instigating a trail of the so-called *retentions,* according to their temporal distance from their Now-moment, when they were primal impressions. And there is also a *just-about-to-arrive* presentification, which is an anticipation that is meant to be confirmed or rejected by subsequent experience, instigating yet another trail, of the so-called *protentions*. The conscious—in the broader sense that Husserl gives to the term—(cognitive) present is the *synthesis* of these three states of consciousness,[19] in what he calls the *flow of appearances* (e.g. Husserl, 1991, p. 126; 2001, p. 51). When I look at the wooden table as I approach it, every Now is simultaneously a *primal perception* of what I actually see this very moment, a *fulfilled anticipation* of what I was expecting to see and an arising of *new expectations* of what I am about to see. How does the perceived object appear in the flow of appearances as a whole, according to Husserl?

> The object appearing constantly new, constantly different, is constituted as the same in these exceedingly intricate and wondrous systems of intention and fulfillment that make up the appearances. But the object is never finished, never fixed completely. (Husserl, 2001, p. 50)

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[17] And it is not necessarily moments but rather like "stripes that have certain 'thickness'" (Husserl, 1991, p. 217).

[18] Presentification is a term used by Husserl in order to distinguish objects of regular perception, which are given directly and with 'in the flesh' here-and-now presence (presentation), from objects that do not have this in-the-flesh givenness (re-presentation) (Moran & Cohen, 2012, pp. 260-261). Re-presentation and presentification need to be distinguished from representation, since not only do they additionally bear temporal features that are pre-constituted and their history is registered (e.g. in recollection, the awareness of their past feature), but they also retrieve their intentional history (e.g. in perceptual acts that are recollected and even in futural presentifications that "can in principle be present"), in gradations of clarity and fusion (cf. presentification in Glossary).

[19] Cf. the Table 1 and the Graph 3 in Part1, §1.5.7, and Husserl's drawing in Husserl 1991, p. 243, which he describes in pp. 242-244; this material originates from Husserl's manuscripts that Edith Stein omitted in her editing of the book on inner-time consciousness (cf. Husserl, 1991, fn. 64 (p. 242), fn. 31, 32 (pp. 223-224)).

The flow of appearances of the table, as I approach it, make up "exceedingly intricate and wondrous systems of intention and fulfillment", and the object is "constituted as the same" although I am aware that in each of these appearances there are always sides of the table hidden from me. Moreover, "the object is never finished, never fixed completely", since there can always be found 'details' that will lead to a closer determination (e.g. features that will enrich my perception of the table, related to its interior, its resistance to certain actions, its functioning etc.).

Husserl was the first to thematise the activity that takes place prior to reflection, which he called *passive stage*, in terms of possibility and necessity (Husserl, 2001), rather than by employing *psychologistic arguments*—i.e. arguments that would allow logical processes to be considered as parts of psychological processes.[20] Having being a mathematician under Weierstrass he strove to ground mathematical objectivities in historical, spatiotemporal lived experiences. The first step that Husserl's phenomenology—the so called *static* phenomenology—makes, is the *constituted identity* of the object that appears "constantly new, constantly different", in replenished "intricate and wondrous systems of intention and fulfillment". And at the same moment the Husserlian phenomenology opens up the object's *indeterminacy* ("never finished, never fixed completely"), occurring from the object's inexhaustibility and the learner's transcendence of meaning, an ever renewed meaning; which is what Husserl calls *determinable indeterminacy* (Husserl, 2001, pp. 42, 43, 45)—which is "an emptiness to be filled-out*"* (p. 42) when it comes to the "transition of appearances" (p. 48)—and Husserl distinguishes it as a primordial[21] feature of perception.

Consequently, there is a tension that is indigenous (immanent) in consciousness' *determinable indeterminacy,* which is the core of cognition according to Husserl. Moreover, "the *formation of unity through coinciding* as it pertains to sense" (Husserl, 2001, p. 48, emphasis added) "also brings, together with the fulfillment, a *closer determination*" (ibid. emphasis added); and it is precisely this

[20] Cf. Index/Glossary for *psychologism*.
[21] And in this sense *universal*.

closer determination that knowledge actually is equivalent to, such as the aforementioned table that became 'a wooden table' and then 'a (wooden) table with a red ball under it'. Whilst *closer determination* takes place in the context of *determinable indeterminacy,* such as the sides of the table that I suppose how they look like although I have not (yet) seen them. Therefore, Husserl's approach to objectification (or *objectivation* insofar as the inaugural stages of objectification are in focus) is characterised by *determinable indeterminacy*, which is 'resolved' by *closer determination,* although through *never fully resolved determinations, due to limitations of the inner and outer horizons of every perception* (horizon as limit-horio-ὁριο).

a. Introduction of the living body and the operative intentionality, from Husserl to Merleau-Ponty

What made Husserl's theory particularly interesting for Merleau-Ponty, who was studying Husserl's work when he died in 1938, without having full access to the vast unpublished work that Husserl left behind and Herman Leo van Breda rescued from Nazi Germany,[22] was not the identity that the object acquires, as the condition for the possibility of its existence as such. Rather, it was that the aforementioned process, with its *determinable indeterminacy,* is not an isolated mind process. *Embodied consciousness* is what Husserl called the *living body* (Leib) or *lived-body,* which became a main theme for Merleau-Ponty, who crucially associated some of his key concepts with it (such as the *motor intentionality* and the *body schema*),[23] and it is a key concept for my study. Here is an introduction to it, using Husserl's own words:

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[22] Herman Leo van Breda traveled to Freiburg, Germany in 1938 for the preparation of his PhD thesis, where he found, in the legacy of Edmund Husserl (1859-1938), more than 45000 pages of hand-written manuscripts, composed in an obsolete form of German shorthand, the *Gabelsberger* system (further modified by Husserl himself to include philosophical terms), and his complete research library, which he moved to Leuven, Belgium. He also convinced Husserl's former assistants, Eugen Fink and Ludvig Landgrebe, to collaborate on the editing of these documents in the university library in Leuven. The university library was burned to ashes on 17 May 1940 but fortunately, one week before the fire Van Breda decided to bring the documents to the Higher Institute of Philosophy (cf. Moran, 2000, pp. 63-64).

[23] Cf. Morris, 2012, pp. 59-63, 54-56 for Merleau-Ponty's *motor intentionality* and the *body schema* respectively.

> We mentioned in passing time and again that the courses of appearance[24] go hand in hand with the orchestrating movements of the lived-body. But that must not remain something that we only mention haphazardly in passing. The lived-body is constantly there, functioning as an organ of perception; and here it is also, in itself, an entire system of compatibly harmonizing organs of perception. The lived-body is in itself characterized as the perceiving-lived-body. We recognize it then purely as a lived-body, subjectively movable and in perceiving activity, as subjectively self-moving. In this regard it does not come into consideration as a perceived spatial thing, but rather with respect to the system of so-called 'movement-sensations' that run their course during perception, in eye movements, head movements, etc. And they do not simply run parallel to the flow of appearances there; rather the kinaesthetic series under consideration and the perceptual appearances are related to one another through consciousness. (ibid.)

For Husserl, "[t]he lived-body is constantly there, functioning as an organ of perception" (Husserl, 2001, p. 50). It is "in itself, an entire system of compatibly harmonizing organs of perception". "[I]t does not come into consideration as a perceived spatial thing, but rather with respect to the system of so-called 'movement-sensations' that run their course during perception, in eye movements, head movements, etc.", and it is not a "perceived spatial thing" (ibid.). The *"perceiving-lived-body"* is not detachable from the 'mind', and Husserl's radical divorce with Cartesian and Kantian perceptions of cognition starts right here, where Husserl conceives consciousness as profoundly embodied. Its embodied character shapes it as *intersubjective, dialectic, temporalising and spatialising* right from the start. As I try to push the table for the first time I'm surprised by how much lighter it actually is than I expected it to be; I realise—in reflection—that my body had already anticipated a certain resistance, without any prior reflection. What appears before any thought arises is an expectation (*protention* in Husserl's terms), as it is the case for the anticipated table's resistance in the table example. What appears

[24] I.e. the flow of appearances.

before any thought arises is an embodied sense that I consciously communicate with from within, without any mind/body dualism, without the "play of the ego", yet with a certain mineness, an awareness that it is my experience.

The *how* of the richness that the lived-body avails for cognition is visible as soon as one removes the 'certainty of the world' and focuses on the 'ambiguity of the world', when one starts to perceive the learning praxis as an achievement and as reactivation (de-sedimentation), under new spatial-temporal and intentional, horizonal thematisations; and the factualities approached as appearances and as material for constitution. For "[a]ppearance is, within me, reality, and the being of consciousness consists in appearing to itself" (Merleau-Ponty, 2002a, p. 439). Merleau-Ponty, carefully read, also summarises in this phrase how the appearance of an *objective* world is transformed to a *world of objects,* under the phenomenological view. Merleau-Ponty's approach to the 'I' of the *cognising* subject meets both the Husserlian 'unthought' (Merleau-Ponty, 2002b, p. 14) and my study's deeper layers of how subjectivity can be approached in a more inclusive and real sense:

> The acts of the *I* are of such a nature that they outstrip themselves leaving no interiority of consciousness. Consciousness is transcendence through and through, not transcendence undergone—we have already said that such a transcendence would bring consciousness to a stop—but active transcendence. (2002a, pp. 335-336).

What is most important for us here is that all these acts and actions are based on particular profiles of the learner's lived experiences, perceived and anticipated (*protented* and *apperceived* in Husserl's terms) through the perceptively orchestrated actualisations of the lived body, which is Husserl's (and Merleau-Ponty's)[25] embodied consciousness. Moreover, Husserl (2001, pp. 228-229) tells us how objects are constituted insofar as they "exercise[s] an affection", and thus they

[25] E.g. Tito, 1990, p. 185.

are "grasped explicitly" through awakening. It is related to the "allure of the objects", which will be detected in the analyses of each of the three case studies:

> Awakening is possible because the constituted sense is actually implied in background-consciousness, in the non-living form that is called here unconsciousness. Here awakening is also the production of an affective communication and therefore the production of a relevant synthesis, of an objectlike connection which, as connection, is actually produced like a simple object, affective for itself. Once something is constituted in the manner of an object, it can connect with anything else that is already constituted as an object; in this instance, precisely two kinds of conditions are to be fulfilled, on the one hand, the conditions of content, of both senses of the objects, on the other hand, the conditions belonging to the consciousness of them. Only something that is grasped explicitly, that is, something that exercises an affection, can become connected in consciousness.

Thus, *awakening* is the other side of *sedimentation*, which is the outcome of *retention*, since "every accomplishment of the living present, that is, every accomplishment of sense or of the object becomes sedimented in the realm of the dead, or rather, dormant horizonal sphere, precisely in the manner of a fixed order of sedimentation" (ibid.).[26] Implicitly existing in background-consciousness *sense* is brought to life, "constituted in the manner of an object". The object is the "production of an affective communication", which leads to a *synthesis* that is *operated* by the lived-body.

And intersubjectivity is a component the subjective act (an intentional horizon), not by psychologistic preconceptions of "how people think", but by *necessity,* since it is the channel through which *sense* becomes *communicable*: "[o]nce something is constituted in the manner of an object, it can connect with anything else that is already constituted as an object".

[26] See Part 1, §1.4.7 for a more detailed approach of retention.

b. Husserl's response to the question of certainty

The visibility of the learning praxis in the Husserlian context—with the Merleau-Pontyian primacy of perception on the foreground—consists precisely in adopting the methodology of the *phenomenological attitude,* whose effects and targets related to my study I just outlined; it is where learning episodes become phenomena in the Husserlian phenomenological sense, where perception and certainty acquire a new perspective:

> [T]he world to which perception 'throws me open' is a *world,* and there is 'absolute certainty of the world in general, but not of any one thing in particular' (*PP* 297/347),[27] because those things 'outrun' both my perception and myself: it is 'absolutely necessarily the case that the thing, if it is to be a thing, should have sides of itself hidden from me', as well as its horizons of past and future; the world 'transcends' my perception, so that perception 'cannot present me with a 'reality' other than by running the risk of error' *(PP 377/439;* cf. *PP 296/345-6).* Truth and the possibility of error go hand in hand. (Morris, 2012, pp. 105-106)

The question of certainty "of any one thing in particular" and of the question of knowledge—the theme of Meno's paradox—in general was raised by Descartes for the first time in the modern era. He inquired back into the ultimate source of all knowledge, doubting the truth of all his beliefs in order to determine which beliefs he could be certain were true. This fundamental doubt (the *Cartesian doubt*) led him to his own thought as the ultimate proof of existence and truth; for even if I doubt for the truth of my senses, I do not doubt of my thought of them, regardless if is illusory; in fact the fallibility of sense data in particular is a main theme of the Cartesian doubt.

[27] "PP" stands for Phenomenology of perception in C. Morris' book; the first number concerns the pagination of the French edition while the second number concerns the pagination of the English edition, which is the same as the one that I include in the references (that is Merleau-Ponty, 2002a).

But if the ground of knowledge is the infallibility of my thought, against my sensuous raw data, the mind/body dualism is already set, accompanied by the hierarchy that the prevalence of the so called *mind* implies; and the concomitant methodology of investigation is not different than the natural sciences, while the validity of the thinking act is founded purely on the alleged timelessness of mathematics, the mathematical sciences and logic, as the ultimate foundation of infallibility, and the guarantee of legitimisation (Husserl, 1970a, §§ 16, 19, 20, 21).

"[W]hat Descartes only wished to establish, and found so hard to establish, by inquiring back into the ultimate source of all knowledge: namely, the absolute metaphysical validity of the objective sciences" (1970a, p. 91). Descartes' starting point therefore is the *cogito,* the 'I think', which is also the instrument of judgment over the *cogitationes,* namely the objects of thought.

Following Husserlian paths Merleau-Ponty announces that "I am not a constituting thought and my 'I think' is not an 'I am'" (Merleau-Ponty, 2002a, p. 437). At the same moment Merleau-Ponty criticises Husserl (cf. fn. 16 in p. 437) and it is already indicative that Merleau-Ponty's radicalization allowed a new life for these ideas, which is also true for many other major thinkers, starting with Heidegger, Derrida, Sartre, Levinas, Gadamer, Ricoeur, and others (Moran, 2000). Consciousness' embodied character and the *operative intentionality* were main attractions of Husserl for Merleau-Ponty, who clarified Husserl's negation of the Cartesian tenet *cogito ergo sum* (I think therefore I am). Merleau-Ponty's investigations attempted to unearth Husserl's *unthought* (2002b, pp. 14, 15), as was the case for Merleau-Ponty's distinction between *speaking speech* or "language in the making" (Merleau-Ponty, 2002b, p. xxviii), and *spoken speech* or "'ready-made' language" (ibid.). This distinction is particularly relevant to my study, since one of its themes is *mathematics-in-the-making,* and the particular—phenomenological—ways that it can be realised.

As I was writing the final parts of this research I read the following extract in an introduction to Husserlian phenomenology, written by some of the most renowned scholars in the area:

> The basic insight that characterizes the idea of authentically genetic phenomenology is then brought forward. This insight is that Husserl no longer speaks of the "I" as empty pole of identity—a position he still held in *Ideas* (1913)—but rather elaborates a concept of "I" which possesses capabilities, takes positions, has convictions, etc., and to which *the world is pregiven as the horizon of an "I can" do this and that*. The task of genetic phenomenology will be one of investigating into the history' of the "I," one of inquiring into the very origin of constitutional systems and the objects constituted within these systems. (Bernet, Kern & Marbach. 1999, p.8, emphasis added)

As the extract clearly states, the essential ("basic") Husserlian insight for [the need for] an authentically *genetic* enquiry—which takes place *besides* his *static* enquiry (Moran, 2012, p. 50)—has the task of "investigating into the history of the "I," one of inquiring into the very origin of constitutional systems and the objects constituted within these systems" (op. cit.). And this authentically Husserlian genetic investigation—according to his genetic phenomenology—can start by expanding on the learner's *life world,* where s/he perceives the world, pregiven as it appears, and most importantly, as the horizon of an "I can" (rather than 'I think'). The operations that took place for the mathematical objects to be realised as such, in the learning episodes that will be analysed, will be elaborated, as thoroughly as the theory allows (and as the students kindly made available).

In this section I gave indications and I implicated necessary terminology that will be unfolded in the following sections, concerning the experimental research that I started and that will be presented here. Concerning the terminology I will once more invoke Husserl's own words (1983a, p. 201, cited in Part 1 §1.1). In other words, the terms introduced in this section will "remain in flux" while acquiring sense through theoretical, methodological and analytical explications, also bearing in mind that "a demand for mathematical exactness of definitions is totally inappropriate in phenomenology" (Husserl, 1970a, p. xxii).

c. Resumè

My research, after my long experience as a teacher of mathematics is a close investigation of the origins of abstract (mathematical) objects, where I explore paths that my instinct (intuition) told me they may release a novel perspective (horizons) of the 'I learn', once a particular lens towards the learning phenomenon is adopted— that is a theoretical lens, which indicates (summons) a fitting methodology. My teaching experience was suspended— although using it in order to understand what the students were doing—attempting to take a close look at the data from their perspective, beyond pre-conceptions of correctness. My natural teaching attitude was put *out of play* as the learners themselves were setting up their objects, hence unlocking the tasks that they were engaged in. My participation is consciously (phenomenologically) reduced; even in the case of Mary my active participation is an echo of a claim that the learner possibly found ready for use.[28] But the acknowledged possible influence is too distant from the theme of my analysis to be taken into account, since my analysis is concerned with the student's subjective (first-person) and embodied intuitive take up of the task at hand, that is the *manner* in which the student (possibly) addressed at home my suggestion in the classroom, even if it was my suggestion that she followed. In summary, the learning phenomenon is analysed in this study for three distinct cases of three prospective teachers of mathematics, aiming at becoming a paradigmatic theoretical, methodological and analytical approach in our field, by indicating forms of action and interpretation of action, of the protagonists of the mathematics classes, namely the learners. And these possibilities for action and interpretation of action are expected to emerge from a deep understanding of learning as a phenomenon, employing Husserlian theoretical and methodological instruments with a new analytical sense.

It is a rare feeling of accomplishment to see at the temporary pause of these paths—that the delivery of this dissertation actually is—that people who have deeply

[28] My suggestion during the classroom investigation was to find more points belonging to the sought curve – see Part 3, § 3.4.2.1.

studied the Husserlian texts 'define' genetic phenomenology as an enquiry with the same motivation (intentionality) as my research has adopted and practiced (Bernet, Kern & Marbach, 1999, op. cit.). Thus inscribing my dissertation to the long and wide stream of phenomenological research that Husserl initiated and envisaged.

My conviction is that introducing an age old yet only recently fully exposed theoretical work of this calibre to our field, from the viewpoint that my research advocates for, may become an instance of *closer theoretical/philosophical study* in the field of mathematics education, and of more *theoretically justified* implications in everyday *classroom/learning in any environment* episodes, as well as an instance and a pole for deeper engagement with Husserlian theoretical ideas. Considering that these ideas may find a path for action and positive apprehension of the learner's needs, towards their approach to mathematics and *mathematical objects* in particular, advocating for an immanent (inherent) inclusive character of the teaching praxis. The detailed elucidation of Husserl's theory of perception that follows in Chapters 2 and 3 (as the space and scope of this research study allow), informed by Merleau-Ponty's primacy of perception, will enable the linkages with the *methodological* and *analytical* approaches to the data sources to be appreciated, as the most appropriate for the chosen theory; and the concomitant *findings* to appear as the elucidation and the exemplification of the theory that supports them. Finally, the explication of the theoretical ground of my research will allow a subtle distinction to be made between the empirical and the transcendental aspects of my study, since my research seeks for empirical evidence whilst being a transcendental study, due to its particular intentional approach to cognition that will be unfolded in the following sections.

d. The aim of the theoretical exploration of the research

Since the aim of the theory section is to explore the theoretical notions that are necessary for the understanding of this research, as a whole and as its parts (chapters, as well as analytical methods), the starting point needs to be clarified. The starting point is also the *aim* of the study as a whole, i.e. the driving force of the theoretical analysis that follows, of the methodology and the concomitant

methods that rendered the data accessible as findings.

The aim is *to explore the emergence of abstract and empirical intuitions and their links to the objectification process, in common and uncommon settings;* and it is complemented by my *motivation:*

- to afford mathematics education research with a better understanding of the learning praxis, due to the phenomenological instruments that the theory provides.
- to enrich the teaching practice with a particular attitude and analytical tools, in order to allow teachers to access their students' ideas for what they could be, as soon as they would be legitimised.
    - By designing accordingly their engagement with mathematics in the classroom, by nurturing intuitions and their corresponding objectivations (objectifications).
    - By developing compasses for teaching practice, due to their phenomenological understandings of the structuring of empirical and abstract (categorial) intuitions, and their links to the students' lived worlds.

In summary, the motivation is to afford a better understanding of the learning praxis, due to the phenomenological instruments that the theory provides, and to look at it under the prism of the complexity of the mathematics classroom (Davis, 2011, 2012; Davis & Simmt, 2003, 2006). It is for this purpose that the critical links between intuition and objectification (objectivation) are investigated, and that their trails are tracked down to the learners' lived experiences.

Given Husserl's focus on lived experience, this effectively reversed the Cartesian method (1970a, p. 171-172) by moving *bottom-up,* from the learner's lived-world and her lived experience, to the abstract mathematical entities; focusing on their emergence from the Life-world, and following their intentional life. This is the reason why the *first person perspective* was chosen, since it allowed to get a glimpse into the silenced world of perception, which resides in what Husserl calls the *passive stage* of lived experience, which will be explored theoretically and

analytically. This rich world of the *passive stage* of lived experience resists discursive communication, while it enriches and furnishes the text that we actually speak, our *speaking speech* (Merleau-Ponty, 2002a, p. xx ; 2002b, p. xxviii). The theoretical exploration of the following chapter will be engaged in the distinction of the Husserlian and the Kantian theoretical frames, the explication of *objects* and *objectivation* (i.e. focusing on objectification's early stages, in the Husserlian sense), as well as *intentionality* and its different and crucial functions for objectification and intuition. And all of it orchestrated by the living body, the embodied consciousness, the body-subject, the empirical ego, which, in its different names and functions feeds the transcendental ego with syntheses of ready-to-be objects. Linking the learning subject with her lived reality, her spatiotemporality that makes objects intelligible in their endurance (Gegenstand),[29] and renders them communicable in their oral, written or other expressions (e.g. gestures).

[29] Gegenstand is one of the two German words that Husserl used for objects, and it etymologically derives from standing-against, interpreted as standing-against the flow of conscious passive perception. Gegenstand has similar etymology to the English and the Greek term, already indicating a cultural ground, since the English term derives from Latin, and the Greek is a rendering of the German term. The other word used by Husserl is Objekt, which is reserved for a more advanced stage of objectification than the one mentioned here. We will see it in the three learning episodes, when each learner's Gegenstand becomes an Objekt of her/his investigation, as its thematisation becomes central for the treatment of the task, thus setting up new and specific intentional horizons on behalf of the student.

CHAPTER 2. Kantian and Husserlian approaches to the cognitive experience – Some clarifications towards contemporary studies that use constructivist or materialist theoretical frames

1.2.1 Kantian and Husserlian approaches to objectification and intuition – Clarifications towards contemporary materialist and empiricist studies that use phenomenological theory

Most definitions and classifications of intuition are endowed with a *constructivist* lens originating in Kant, Descartes or Plato, especially when it comes to abstract intuitions. Processes that inhabit the *cogito,* the discursive human mind, take over, and the abstract intuitive processes are ultimately settled in internal, relational cognitive models. The mature Husserlian approach on experience (after his Time Lectures and Ideas II)[30] that my research adopts, cannot be properly understood as linked to a meta-Kantian philosophy without a space permitting clarification of its crucial yet highly neglected divergence from Kant's notion of *experience* and *knowledge.*

1.2.2 A critical difference between Kantian and Husserlian approaches

"Experience is an empirical knowledge," writes Kant (cited in Tito, 1990, p. 78). By experience Kant means *objective* thinking, *knowledge.* In other words, there is a predisposed categorial network that the subject is stored with, which makes perception possible. A general remark of the issue that concerns us here, as we trace the origins of what makes cognition possible, is that the difference between Husserl and Kant is very subtle yet crucial, since Kant recognises experience as *already* endowed with objectivity, while Husserl re-cognises experience, by considering objectivity itself as being *constituted* by the subject(s) (as a major pole of constitution), the communities and the social-cultural-historical entities, while the

[30] Husserl, 1991 and 1989 respectively, in the references.

scientific communities are also at the centre of his interest (1970a).

Husserl introduces "a new phenomenological attitude which focuses on the correlation of experienced and experiencing within lived experience itself as a foundation of the sciences" (Husserl, 1983b, p. ix). In a necessarily crude sense we could say that for Kant, the objects that are constructed by the subject are 'separations' of experience by thinking, which is already categorial; while for Husserl "[t]he object is the idea of the fulfilled sense (as fulfilled completely)" (Husserl, 2001, p. 449). Which leads him to the essential "how the 'objective' a priori is grounded in the 'subjective-relative' a priori of the life-world or how, for example, mathematical self-evidence has its source of meaning and source of legitimacy in the self-evidence of the life-world" (1970a, p. 140). Kant's persistent categorial membrane separates the subject from the world by making impossible her immediate contact with it, while Husserl introduces the immediacy of perception and objects as "self-given in a simple intuition" (Husserl, 1973, p. 150, §12).

Kant distinguishes sharply between *sensibility* and *understanding*, which is already structured categorically. For Husserl there is continuation, similarities and structural as well as genetic continuities between the two, explicating 'I think' from 'I can', 'I could', from the potential and the abilities of intentional consciousness—*potentiabilities* is the term that Husserl uses—operating within and reflecting on the life-world. A life-world that remained unexplored before Husserl's mature work, who constituted it as a term, a world lived through by the individual, from the proto-geometer (Husserl, 1936/1989, p. 87) to the proto-learner.

Kant's principle idea that the transcendental ego is already installed with some qualities *beforehand (a priori*) remains as a *genetic* seal, inherited as we shall see by the constructivist followers of Kantian ideas in mathematics education and in particular intuition. Moreover, Kant's concept of intuitions does not anticipate any abstract intuitions, apart from the a priori intuitions of time and space. And these intuitions yield knowledge due to their ideality and not due to their immediacy (Hintikka, 1972, p. 344). Husserl's approach is in sharp contrast to Kant's approach, since he argues for the constitution of all categories through his concept of *categorial* intuitions (cf. Part 1, §1.5.5); therefore Husserl's innovation is the

abstract intuitions and their structural and factual links to the empirical ones, due to the thread of the empirical ego, which he called *living body*, (e.g. Husserl, 1970a), and Merleau-Ponty (e.g. Merleau-Ponty, 2002a), after Husserl, called *lived body*.

But before delineating these concepts it is necessary to present Husserl's account of knowledge itself and in particular how new knowledge is possible. This has obviously consequences to the question concerning the possibility of new knowledge that is alien to previously objectified knowledge (which is the core of Meno's paradox). Husserl's answer to this question is related to new cultural horizons that are grasped originally (originarily) by the cognising subject, which is intentionally driven towards objects and intuited states of affairs due to what is at task, for the task at hand. The learner gets to new territories of so far alien objects and object-like formations—which could be read as an evolution of the Aristotelian morphè(see Held, 2010; Luft, 2007; Niel, 2010; Zahavi 2010).

It unfortunately requires the space of a proper publication, without the current word limitations in order to provide an account of the innovativeness that Husserl has inspired to contemporary mathematics education research approaches, most often without addressing Husserl's contribution directly (De Freitas & Sinclair 2012, Radford, Roth). Empiricist and materialist approaches to learning that make use of phenomenological perspectives (e.g. De Freitas & Sinclair, 2012, 2013; Roth, 2009, 2010, 2012) offer a rich source of ideas, although avoiding or reducing the transcendental aspect of knowledge. Both my research and the aforementioned ones explore the cognitive praxis as essentially embodied, and explore it beyond the 'I think', towards the 'I can' and 'I could'. But the main difference occurs from my focus on the cognitive praxis as intentional, in the Husserlian theoretical context that will be unfolded in the following sections. By distinguishing theoretically and analytically intentionality into *operative intentionality* and *intentionality of act*, and by employing the Merleau-Pontyian reading of Husserl on lived experience, through the lived-body, I will manifest Husserlian concepts as operational for learning situations of everyday classroom or other cognitive experience. And these concepts will be put to practice in the analyses of the three learning episodes (Part 3). Although my principal aim in this study is to clarify important theoretical distinctions

between the Husserlian and the Kantian/constructivist perspectives to understanding mathematics, I could summarise the main theoretical/methodological distinctions between my study and the materialist phenomenological ones that I mentioned occur from two main sources, concerning my methodological approach:

- My Husserlian approach on subjectivity, "purely as intentional, that is, as transcendental" (Carr, 1999, p. 97). And the uncovering of the layers of intentionally lived cognitive life, corresponding to less or more epistemic—in our case mathematical—objects and layers of *constituted* objectivity. The critical difference with materialistic phenomenology is the *immanent transcendence* of the *constituting* subject that Husserl's transcendental theory allows, when properly understood. Husserl's theory puts the achievements of *immanent transcendence* at the centre of his investigations, which is also an implicit layer of my analyses of the learning episodes (Part 3). Husserl's radical response to Kant's approach to the cognitive experience is based on intuition, as the original source of evidence in the life-world and as the ultimate ground of *objective truth:*

  > The supposedly completely self-sufficient logic which modern mathematical logicians *[Logistiker]* think they are able to develop, even calling it a truly scientific philosophy, namely, as the universal, a priori, fundamental science for all objective sciences, is nothing but naivetè. Its self-evidence lacks scientific grounding in the universal life-world a priori, which it always presupposes in the form of things taken for granted, which are never scientifically, universally formulated, never put in the general form proper to a science of essence. (1970a, p. 141)

- The actual implication of phenomenology purely as method (cf. Carr, 1999, pp. 96-97; fn. 182 in Husserl 1936/1989), focusing on the intentional forces that operate in pre-reflective and pre-objective consciousness, while using successive reductions and bracketing as a mechanism in the *zig-zag* movement (Moran, 2000, p. 93) between the theory and the data/data

sources. In other words, the process where the flow of data are transformed to findings due to the phenomenological methodology, and the application of the latter to the findings—i.e. to the products of the analysis—entails further reflections and analyses, which yield further findings (see Graph 4, in Part 2, § 2.2).

1.2.3 Clarifications towards contemporary materialist and empiricist studies that use phenomenological theory frames – A further requirement for Meno's paradox and the introduction of the Husserlian *principle of all principles*

Let's take an example, concerning the way that Roth (2010, pp. 8-9) depicts the problem of Kantian-based constructivism and its radical ramifications, in relation to Meno's paradox. Roth refers to a case where second-grade students were called to identify shapes of objects due to solely tactile maneuvers, amongst them a cube. It is a tactile-oriented and similarly analysed cognitive 'moment', while aiming at projecting the prevalence of the tactile sense over all others:

> How can the mind intend constructing a cube when it does not know what a cube is and therefore cannot intentionally aim at and construct it? This is the dilemma that Kant, the 'logodedalus' (Nancy, 2008), the master artisan of the mind, has never solved. But then how did we come to understand that by looking at or touching some solid square faces we were seeing / touching a cube?
>
> The problem for the intellectualist mind is known in the cognitive sciences as the 'symbol grounding problem,' that is, the problem how mental representations are related to anything other than mental representations (Harnad, 1990). How does mind know that what it has constructed relates to anything in the world? Because social constructivists (e.g., Cobb, 1999) cannot answer this question, they have to assume everything to be 'taken-as-shared.'

Concerning "[h]ow can the mind intend constructing a cube when it does not know what a cube is" one could for instance track the second-grade students' saccades

and fixations (using an eye-tracker), thus tracing visual and *kinesthetic* activities running parallel to their tactile ones. Perhaps one could support the claim that the tactile prevalence is a convenient metaphor, while kinethesis is at the base of all senses, amounting to Merleau-Ponty's sensorimotor intentionality. But reading according to our paradox orientation, we notice in the *'symbol grounding problem'* terminology that Roth uses another *paradox* and its relation to Kantian presuppositions, concerning the very *contact* of our *"mental representations"* with "anything in the world".

Expressed in a Husserlian/Derridian gloss, the Kantian questioning that remains hanging in mid-air concerns how does our mental *presence* (our *understanding* in Kantian terms) is related to the transcendence of *sensible* objects (their inexhaustible appearances), and to the constituted *idealities* such as the mathematical ones, deriving from geometry, analysis etc. and abstracted by the categorial mind, before any interference of experiments that put to the test our cognitive certainty. That is, how certainty does emerge immanently, fed by the transcendence of the subject's lived-world and cashed in by transcendental subjectivity, before and as it becomes a conviction. So the question that Roth asks is at the heart of the Kantian problematique, and from my cognitive point of view it unravels the constructivist programme and its origins. Roth, expresses the same idea as he describes the dead-end of the Kantian-based constructivist programme, as he continuing the previous extract:

> In Kant's constructivist approach, the knowing subject and the object known are but two abstractions and a real, positive connection between the two does not exist (Maine de Biran, 1859a, b). The separation between inside and outside, the mind and body, is inherent in the intellectualist approach whatever the particular brand. This is why Ernst von Glasersfeld's (*e.g.,* 1989) radical constructivist mind knows nothing about the world and remains stuck in its own representations that are tested for fit. (ibid.)

The question that Roth refers to in the aforementioned extract, put in Kantian terminology, is how the transition is possible, from *experience* to the *abstraction* of

experience*,* when *sensibility* is divorced from *understanding* as two different faculties of cognition, in their *distinguished complementarity* and *equal necessity* (Kant, 1998, A51, B75), thus linked externally to the latter? Doesn't then testing become the sole guarantee of our mental constructions, such as those anticipated by constructivist mental representation frames? And does not cognition start only after the *reflective* mental activity, making use of intuitions in an evolutionary, biologistic sense (Fischbein, 1994, 2002; Beth & Piaget, 1996). This constructivist approach is an external (non-cognitive) solution to the aporia of Meno's paradox, since the subject is moving towards new knowledge being afforded with a categorial mind beforehand, and 'forced' to construct new knowledge due to her evolutionary biological nature. The newly appearing schemata, drawing their motivations from the learner's involvement in the world of experience involve *categorical* configurations tested in action before they are acknowledged as new. In other words they have a missing link in the chain to the real, to the raw lived-experience of the world-as-lived, since the process that leads to this 'acknowledgement' is taken for granted.

In the same manner, Kant and Kantian-based theoretical frames have an indirect link to the raw lived-experience of the world-as-lived for the following three reasons that are related to this study:

- Kant and Kantian-based theoretical frames perceive *sensibility* as separate from *understanding*, and they are working on sensibility as ready-made for understanding. In other words, although for Kant and Kantian-based constructivism "even in the case of mathematical objects, we need the support of intuitions, that is, of particular representations of the objects" (Radford, 2002, p. 223), the reduction of intuitions to "representations" is crucial, particularly since these 'representations' are not questioned or dealt as re-presentations[31] and the absence of abstract intuitions apart from the a priori intuitions of (Euclidian) space and (Newtonian) time that are

[31] For the use of re-presentation in Husserl (distinguished from that of representation) see fn 17 in Part 1, §1.1, and re-*presentation* and *presentification* in the Index-Glossary, at the end of the study.

somehow installed has led to the *reduction* of *understanding to a construction setting*.

- Continuing the previous distinction, comes the distinction between the appearance and the in-itself of an object. Luft's (2007)[32] words, give an adequate description of the inaccessibility of the thing-in-itself, the essence of an object, and the rejection of this conception by Husserl.

> Husserl rejected what was precisely the main tenet of Kant's transcendental idealism, i.e., the distinction between thing-in-itself and appearance, as 'mythology' …from the standpoint of phenomenology, a distinction between a thing-in-itself, to which we have no access and about which we can know nothing, and its appearance, of which we have experience and knowledge through our cognitive apparatus, makes no sense. With his distinction, Kant might have opened the door to a 'science of appearances', phenomenology, but the very distinction is a mythical construction. (Luft, 2007, p. 369)

This mythical construction that Husserl attributes to Kant's perception of experience sets the beginning of a 'science of appearances', phenomenology, and an alternative way to approach lived reality. This new way, is congruent with acknowledging the *intentional* character of conscious acts, in reflective and pre-reflective experience; it seeks to describe a 'pregnant' striving for fulfilment of latent and prominent intentionalities. And it

- Kant and Kantian-based theoretical frames are 'blind' to intentionality, and in particular its *operative* functioning, on which the positing, constituting *intentionality of act* is grounded, and due to which the latter is actually possible. This disregard is symptomatic of the different approaches to lived experience that Husserl, and Kant, including and his constructivist heir

[32] Luft's article is arguing towards bridging Husserl and Kant, therefore this quotation has an added significance.

The Kantian/constructivist approach reminds me of Alexander's resolving of the Gordian knot, by loosening it and removing the sticks. The problem seizes to exist although remaining unsolved, from the cognitive point of view; rather, it is a 'solution' that conceals the resources of the actual solution, a 'solution' that cancels any solution whatsoever. The question that Kantian-based constructivism fails to handle in depth concerns a deeper layer, concerning the tension between structure and genesis. But we will not focus in this study on the deeper theoretical issues related to contradictive statements, originating in unclear theoretical originations that constructivist Kantian-based or naively bracketed research in mathematics education is abundant of (e.g. Cobb, 2002). And it brings about the constitutive aspect of cognition and its possibility to meet the real world in the Kantian sense; if Meno's paradox, concerning the (possibility for the) acquisition of new knowledge has any chance to find an answer in the "intellectualist" paradigm, the aforementioned problem is critical since, if it would receive a negative answer then the subjective mental constructions would remain mythical figures, imposed by predisposed categories, and unable to relate "to anything other than mental representations". In case the mind cannot "know that what it has constructed relates to anything in the world", the question of the acquisition of new knowledge remains a separated issue from the real-lived world, while the cognising subject "remains stuck in its own representations that are tested for fit" (op. cit.). In other words, when the experience of the lived world concerning the subject's *sensibility* is strictly separated from the process of *understanding,* as it is the case for Kant and constructivism (e.g. von Glasersfeld, 1980, 1984, 1989), when intuition can only give us particular singularities (e.g. Radford, *The object of representations: Between wisdom and certainty;* 2014), and present them to the categorial net of understanding, with the latter being responsible for the categorial determinations, externally related to the sensuous realm, then the possibility of meeting new knowledge concerns a *mythical world*.

The same question is raised in Husserl's work, as Derrida articulates it: "how can subjectivity go out of itself in order to encounter or constitute the object?"

(Derrida, in Husserl, 1989, p. 63). And for Roth, as well as for my study,[33] this is the case where intellectualist views have failed to grasp the critical cognitive requirement, which inevitably led them to several instrumentalist views of the activist requirement of Piagetian-based theorising of cognitive experience. Neo-constructivist views attempted to link/integrate von Glasersfeld's instrumentalist approach to cognition with different theoretical approaches, such as social interactionism, situated learning and Vygotskian perspectives (e.g. Bauersfeld, 1992; Cobb, 1994; Cobb & Bowers, 1999; Cobb, Boufi, McClain & Whitenack, 1997; Cobb & Yackel, 1996). The ontological commitments to the Piagetian cognitive pedagogy were only bracketed (Davis, Sumara & Kieren, 1996, p. 156, p. 168), in the researchers' intension to release the possibilities for expansion of the Piagetian claim for *cognition through action*.

According to Roth and to my theoretical viewpoint, as I roughly prefigured it in the previous paragraph, the questions that Roth posed remain a puzzle for Kant and his constructivist successors. My work will go a step further towards Husserlian phenomenology, which responds with Husserl's abstract (*categorial*) intuitions, a particular kind of the latter, called *intuition of essence*, and their links with the *empirical* intuitions *of something individual* (1983a, §3, p. 10). As a preface

> [I]ntuition of essence has as its basis a principal part of intuition of something individual, namely an appearing, a sightedness of something individual … no intuition of essence is possible without the free possibility of turning one's regard to a 'corresponding' individual and forming a consciousness of an example—just as, conversely, no intuition of something individual is possible without the free possibility of bringing about an ideation and, in it, directing one's regard to the corresponding essence exemplified in what is individually

[33] Although a reservation may be useful here, since Roth's expression "the intellectualist approach *whatever the particular brand*" (emphasis added) seems to include Husserl, overlooking the Husserlian influence on Merleau-Ponty, even at his latest writings (e.g. Lawlor, in Toadvine & Embree, 2002, Ch. 10, p. 204), from where Roth is drawing upon.

> sighted; but this in no respect alters the fact that *the two sorts of intuition are essentially different.* (italics in the original)

Husserl describes an immanent transition, a lineage from the *intuition of something individual* to the *intuition of essence* (and vice versa), thus introducing a path towards abstract intuitions. Furthermore, he considers the links between empirical and abstract intuitions as indispensable, considering the intuition of something individual as in-itself giving space to the "free possibility of bringing about an ideation". "[D]irecting one's regard to the corresponding essence exemplified in what is individually sighted" brings abstraction and sensuous embodied data closer than ever before, while dualities such as 'empirical' and 'abstract', or 'embodied' and 'mental' work for the transition rather than the separations that these dichotomies indicate in Kantian-based frames. Since the empirical (individual) objects become material for the abstract ones Husserl provides a strong indication for the emergence of abstract objects as *interrelated to individual objects,* thus prefiguring the ground, the source from where subjectivity will go out of itself and meet "anything other than mental representations".[34]

Summing up, the critical difference between constructivism and Husserlian phenomenology amounts to the distinction between phenomenology and phenomenalism, as it is acutely picked up by Hintikka (1995, p. 83):

> What phenomenalism holds is that we have access only to phenomena, not to the real things… What a phenomenologist like Husserl maintains is that everything must be based on, and traced back to, what is given to me in my direct experience. (Hintikka, 1995, p. 83)

In other words, contrasting to previously mentioned Kantian-based theoretical approaches, which have dominated the field of mathematics education in various versions, Husserl's approach to perception in general and intuition in particular already prefigures a response to the question posed earlier by Derrida, i.e. "how can subjectivity go out of itself in order to encounter or constitute the object?"

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[34] Instances of intuition of essence are analysed in Part 3 (§§ 3.2.2, 3.2.3, 3.4.2.3.4).

'Mental representations' are only a 'convenient expression' that is actually very misleading, and "signifying nothing" (Hamlet); quite on the contrary, we need to look closely at the complexity of representations (as re-presentations) and the ground of the learning praxis. Although it transcends the scope of this study, I would be tempted to suggest—as the topic of future investigation—the common ground with the Hegelian dialectic, of the double move that Husserl introduces in the aforementioned extract (1983a, §3, p. 10), from *abstract* to *concrete* and from *concrete* to *abstract*. Both paths will be explored from a Husserlian/Merleau-Pontyian perspective in the following sections, theoretically (Part 1 §§ 1.4, 1.5) and analytically (Part 3). But I'll introduce Husserl's *principle of all principles* here, as an epistemological principle for Husserl, indicating the grounding of science and mathematics on direct/immediate access that is achieved through intuition, in the embodied, conscious, lived-through actuality. According to this principle,

> every originary presentive intuition is a legitimizing source of cognition, that everything originarily (so to speak in its 'personal' actuality) offered to us in 'intuition' is to be accepted simply as what it is presented as being, but also only within the limits in which it is presented there'. (Husserl, 1983a, p. 44)

The principle of all principles concerns the "grounding of all cognition of matters of fact by experience" (ibid.) for Husserl, that is *lived-experience,* which lends itself to the giveness of the real, thus shaping a preliminary answer to the riddle introduced by Roth (towards Kant) and by Derrida (towards Husserl), namely how can "subjectivity go out of itself in order to encounter or constitute the object". Through embodied perception the contact with the raw lived material is made possible and it is precisely this contact that is the sole guarantee[35] that what we constitute does in fact "relates to anything in the world" (op. cit.).

As we will see in the following sections the same principle will become the starting point of unfolding the Husserlian answer to Meno's paradox, which will be formally introduced in the following section.

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[35] It is the case when the contact, as a *phenomenon*, becomes part of our field of critical investigation.

CHAPTER 3. Objectification, the "red book" example, and the employment of the *transcendental phenomenological reduction*

1.3.1 Husserl's notion of objectification – Introduction of the Husserlian approach to Meno's paradox

A brief and focused unfolding of Husserl's theory of objectification will follow, which is considered necessary for the grasping of the objectifications—i.e. object constitutions in a broader sense, including the shifts of the cognitive horizons that the objects entail—that are analysed in Part 3.

The concepts and theory used in this research are drawing on Husserl's theory of objectification/objectivation, as it is explicated in his mature work and posthumously published manuscripts (Husserl, 1970a, 1973, 1983a, 1983b, 1991, 2001), as well as the use of Husserlian ideas (operative intentionality, living body) by Merleau-Ponty (1964, 2002a, 2002b).

Husserl delineates a complete course of the objectification process in five stages (Husserl, 1936, pp. 163-165), from the "intrapsychically constituted structure" (p. 163) of the *proto-geometer,* to the final, ideal stage of the object, through the social scale of "the community of scientists as a community of knowledge living in the unity of a common responsibility" (ibid. p. 169). The *proto-geometer* is not Euclid, Pythagoras, Thales or any other particular historical figure *per se*; the concept of the *proto-geometer* transcends Euclidean geometry, signifying any person who managed to instantiate a property within mathematics and who communicated it to others, up to the point that the new object became independent of its origination and part of the body of mathematics, which is the final stage of the object, one that Husserl terms as *ideal*. The properties of particular kinds of triangles, circles, flat and three-dimensional shapes have formed an axiomatic structure and method of knowledge. This is the "ready-made geometry", the tradition of geometry (Husserl, 1936, p. 158) that Husserl delves into its origins, using it as a paradigm for all science (e.g. Husserl, 1936, p. 159). In other words it

is an alternative historical task that Husserl undertakes, into the (intentional) origins of the object, as it is constituted by the subject, before its social circulation constitution

During the *first stage* of objectification an object appears in front of consciousness, "in original 'self-evidence'" (ibid., p. 163). In the second stage the same object appears and it is identified as the same. Then, in the third stage the subject has isolated an idea of object and applies it to different situations. In Husserl's 4th and 5th stage of the objectification the idea obtains the status of an "absolute ideal Objectivity" (Derrida, cited in Moutsios-Rentzos, Spyrou, & Peteinara, 2014, p. 30).[36] The objects that have arrived at the 5th stage are allegedly untouched by empirical or abstract counterexamples. Although Gödel cautions us that this 'joyful absolute' is inconsistent, we keep on practicing mathematics without being able to notice any impact outpouring from this inconsistency. The issue is definitely far richer than this short reference allows to be exhibited, but it would be an unnecessary deviation to enter the field of mathematics revolutions, which are "at once the least destructive and still the most fundamental revolutions" (Grabiner, 1986, p. 212). Since, mathematics-in-the-making, being the learning patent that this research argues for, is concerned primarily with *constituting* objectivities and the renewal of the pre-constituted objectivities by the acknowledged and legitimised lived experience of cognising subjects, who are themselves carriers of this renewal. It is not concerned with *copying* iconic, ready-made objectivities; it is the relation to the mathematical tradition that is critically approached in this study, explored from the perspective of the intentional horizons that new knowledge springs out. Moreover, from a practical viewpoint the reactivation of *concepts/ mathematical objects* with an embodied startup, is also the theme of this investigation, as one of the studied cases (Mary) will manifest.

Husserl's unexplored work on objectification that concerns us here is related to the first stage of objectification, "the history of the object itself as the object of

[36] A following section is dedicated to objectification/objectivation. All that is necessary at this point is a generic approach of the term, as *constitution* of objects, from the point of view of this study, by the cognising subject.

a possible knowledge" (Husserl, 2001, p. 634). Husserl introduced *objectivation*—his special term for the early stages of objectification—of objects, from their objectlike to their object status (2001, part 3). The object appears for the first time in consciousness as a separation from its hyletic (raw) empirical material, by fulfilling the intentionality that was aiming at it. The fulfilment of intentionality leads to the *constitution of the object*, namely to a novel thematisation, a unity of previously latent features that perception brings together as a profile of the subject's lived reality concerning the object, a new configuration from the 1st person perspective.

The acts of *constitution* don't just appear *in a vacuum,* since:

- The individual involved in the primal constitution of a (mathematical) object for herself is already in a world, which houses and provides the material for the learner's intuitions.
- The subject herself has pre-reflective operative intentionality to render her proto-objects intelligible for others right from the start (Crisis, §54b) and then objects appear in front of her consciousness as existent for others.
- There is a meaningful tension in the interplay between the subject that exercises the objectifying act, and the object that is intended (Kolen, 2005), a negotiation of meaning through the task/activity.

In an *educational context,* the key ideas that were involved in the objects' genesis are 'awaiting' their reactivation, for a meaningful sense of the teaching/learning praxis, and the students are guided through the unfolding of hundreds and thousands of years of evolution that each mathematical object conceals in its refined, 'final', ready-made mode in which it is presented. This aspect is stressed in the cultural-historical approach towards objectification, which is introduced as the process when "students gradually become acquainted with historically constituted cultural significations and forms of reasoning and action" (Roth & Radford, 2011, p 48). In this approach a new object appears in subjective consciousness as a result of learning, by the "transformation of cultural objective knowledge into an object of consciousness" (Radford 2013, p. 25).

The important problem here is that the cultural objects are originally

unknown for the learner: "How will you set about looking for that thing, the nature of which is totally unknown to you? Which, among the things you do not know, is the one which you propose to look for?" (Meno's paradox as it is articulated by Merleau-Ponty, 2002, p. 431). From the phenomenological perspective "perceptual consciousness is not an empty box, into which a perceptual object shows up unannounced and ready-made" (Husserl, 2001, p. 606), but due to the "continual coinciding of sense" (ibid.) new objects appear.[37] And it is the *history of objectivation* (2001, p. 634) that reveals the origins of the *reactivation* of mathematical objects and objectivities in the acts of new constitutions, regardless if the latter have already entered the mathematical domain or are *illuminations* (Hadamard, 1954) that could achieve the final stages of the objectification that Husserl (1936) delineates. Perceptual consciousness is full of object-like formations and may perceive the previously unknown as whole and as true, as long as its appearances hold and can be visited "again and again". But also it is not a box, since the lived-experience itself imposes the rules of *confirmation, identification* and *cancellation-illusion,* through consequent refutations and proofs, proofs and refutations (Lakatos, 1978). Perceptual consciousness is the dialectic ground where objects surface and previous objects are revisited anew, under different (intentional) horizons.

At the moment of constitution the learner manifests herself as the Husserlian *transcendental subject*[38] "that intentionally engages the world and discloses its significance" (Drummond, 2007, p. 204). The constitutive moment reveals both, the constitutive subject and the constituted object, through "the correlative intentionalities of which they are the poles, through whose function they have, and have attained, their ontic meaning" (Husserl, 1970a, p. 182). The intentional identification of the object by the objectifying act, performed with the *immediacy*

[37] See also Part 3, Ch. 3, § 3.1.1 for the development of the argument that is presented here, just before the data analysis.

[38] A following section is dedicated to the explication of transcendence in Husserl, and to cognition as the tension between *transcendence* and *immanence*. The just mentioned tension, located at the core of cognition, is directly linked to Meno's paradox and to the answer that my research adopts, following Husserl and Merleau-Ponty.

of self-giveness is intuition itself, which will be at the centre of the investigation in this research. And we will start with the primordial stage of intuition, by exploring its environment in *passive synthesis*, where an incessant course of acts is performed, and a continual synthesis of identification takes place, proliferating consciousness with object-like entities, preparing the ground for the emergence of objects, i.e. for objectivation, and making objectification possible, up to its final, 'ideal' status (Husserl, 1936).[39]

1.3.2 The Husserlian approach to objectification that is adopted by the research. The example of the "small red book"

The intuitive *recollection* that will be analysed in the following section (Part 1, §1.3.3) concerns an example found in the literature that is relevant to my research (Radford, 2003), and it holds no immediate relevance to Meno's paradox (Part 1, § 1.3.1). But it gives an account of how, even when the intended object for the task at hand is a separation of suitable features from previous encounters with a familiar object of sensuous perception—like a book that was previously read extensively—there are certain implicit acts involved, which go unnoticed without the particular Husserlian phenomenological filter that is applied in my study.

The analysis will reveal how a sensuous object of our life-world and our straightforward involvement in it may yield appropriate thematisations of its features, due to the particular intentions that bring it to the surface of consciousness, as an *intended* object. The analysis will thus reveal the *phenomenological* character of the objects studied here, as they are intended, in contrast to the objects as they appear ready-made in the natural attitude (objects *simpliciter* in Husserlian terminology), which are bracketed here, in order to be better understood. Since the aim is to explore *how* abstract mathematical understanding emerges, through objects that the learner constitutes in her pathway through tasks that she encounters. The phenomenological analytical novelty that

[39] This 'ideal' status is spatiotemporally constituted and it is not a Platonic entity (cf. Husserl, 1983a, pp. 41-42). The issue of Husserl's alleged idealism and my perception of his transcendental idealism will be addressed throughout the study.

takes as its starting point the cognisant's lived experience, and seeks to explore the intentional origins of the objects employed for the task at hand is matched with (the unfolding of) the methodological novelty that makes possible the operation of the aforementioned analytical approach.

The theoretical analysis of the literature example will set the ground for discerning a gap between the Husserlian and the Kantian approaches to cognition that remains unnoticed in the mathematics education research field. This gap concerns the notion of *experience,* and in particular the *objectifying/intuitive* experience; and it has critical effects on the approach to objects, as *constituted* (Husserlian approach) or *constructed* (Kantian approach). The distinction is crucial in identifying the novel approach that my research adopts towards *intuition*, as an *essential objectifying act*. This approach, that will commence at the following sections, will become the main theme of my argument concerning the introduction of a Husserlian approach towards the *commencing stages of objectification* (*objectivation*)*,* which already started with my reference to the reversal of the Cartesian method (Part 1, §1.1.b, §1.1.d)—i.e. the bottom-up process of the analysis, as soon as the intentional origins of the objects are unraveled. And the theoretical exploration will be complemented with the unfolding of the *constitutive* approach that my research adopts, towards the *embodied* conscious acts of 'I can', 'I could'—rather than 'I think'.

Such acts are distinguished theoretically and analytically from the *constructivist* Kantian heritage of *knowledge,* as separated from *sensibility*—a distinction prevalent for decades in mathematics education—which perceives experience in Kant's *mythical* way (Husserl, 1970a, e.g. §30, p. 199). The necessary literature for the issue at task, namely *intuitions* and their *objectifying potential and abilities* (*potentiabilities*) is investigated, revealing the same preoccupations, strict separations and misconceptions. A radically different approach, based on Husserl is offered, one that takes into account the passive, silent and silenced stage of pre-reflective, operative, cognitive process, the ground of all cognition for Husserl (Husserl, 1970a, p. 124), and the essential *groundless ground* (Gachè, 1997) of his Life-world, "for theoretical truths" (Husserl, 1970a, p. 124). But how does the

groundless ground yield *objects,* which populate our conscious life as something 'taken-for-granted'?

The answer advocated for in this study is related to 'moments' when the Life-world comes to life, in constitutive moments that new configurations and modalities enter, tied as they are to the real, being part and parcel of it, yet never allowing the full view. Yet the full view is always anticipated (protented), put to the test for becoming knowledge already existing, never accessed previously, or simply an illusion. In any case, a fundamental intentionality towards what is not seen, towards what is not perceived but co-presentified as being-there is *immanent* to the present perception or, more precisely, to the *perception of the present*, necessarily implicating its temporally modified retentions, the retentions of the 'now' moment(s). So a few clarifications concerning the critical difference between Kantian and Husserlian approaches that concern my study are called for, before entering the analysis of the example itself, and the theoretical investigations that are expected to endow it with new horizons. And Meno's paradox will be introduced, while an allusion towards its unriddling will be indicated.

1.3.3 An introduction to the perspective that this study has to perception – The "image of a small red book"

The intuitive recollection that will be analysed in this section holds no immediate relevance with the Meno's paradox. But it gives an account of how, even when the intended object is a separation of suitable features from previous encounters with the object of perception, according to the task at hand, there are certain implicit acts involved, which go unnoticed if a Husserlian phenomenological filter is not applied. Louis Radford in his article *Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization* (2003) "offer[s] an exploratory investigation of presymbolic types of generalization" (p. 38) drawing from Vygotsky's work and from phenomenology (p. 39). He gives an indicative example on the objectification process from everyday experience:

> Let us imagine that we are standing in front of a wall covered by shelves full of books without any particular intention to look at them. The books all look

> quite similar. Let us also suppose that, although we were looking at the books in this disinterested way, we suddenly remembered that we needed to check something in Aristotle's *Poetics.* The image of a small red book comes to mind. The non-reflective perception with which we began now gives rise to an intended perception. In scrutinizing the shelf, our attention will focus on some red books and, in practical terms, we will almost ignore the others. Suspecting that we will later need to find the same book, we decide to put a mark, or a sign of some sort, on the shelf so that the next time we enter the room the sign will mean something like, "Here's the book!" This mark, or sign, is achieving a particular task: In an elementary way, it is accomplishing an *objectification.* (p. 39, italics in the original)

For Radford the objectification process is linked to a sign ("a mark, or a sign of some sort, on the shelf"), which we use in order to facilitate our search the next time, a sign which "mean[s] something" in its objectification functioning.[40] For my study, the "image of a small red book [that] comes to mind" is *already an objectification,* and a structurally significant one, since it may reveal the unseen mechanism of the objectification process.[41] I consider it an objectification not because of the attribution of redness as yet another sign to the task of finding the book, but merely because the image *is* a mental object, precisely at the moment that it emerges in consciousness, called by our *intention* to be attentive for something particular, that is the sought book's material existence.[42]

Intentionality for Husserl is "the being of the intention" (Origin, 1989, p. 139) and it is understood as the web of intentions that arise, merge, point to different directions as they exist in their similar yet distinct sets of horizons. When we focus on the *material* aspect of the object which, although absent we know that it exists there, on the "wall covered by shelves full of books", we trigger the image of the red book to appear, as the *fulfilment* of our *intention* to substantiate what we are

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[40] The essential links between *objectification* and *meaning* are addressed in the following section.

[41] See the next section and the evidence acquired from empirical data, displayed in chapter 3.

[42] See also Radford's description of objectification in Radford, 2003, p. 40.

looking for. In other words, the task "to check something in Aristotle's *Poetics"* includes another task that concerns the concretisation of what we are searching for; and the latter task puts the mechanism of intentional consciousness in motion, bringing the image of the red book to the surface, in the still unfulfilled ("absent" in Husserl's terms) intention of finding the actual book. Our attention is not enacted *only* when we "will focus on some red books" while "we will almost ignore the others" but, quite on the contrary, this attention is *possible* only due to a previous stage of attention, the one that was directed towards the image of the sought book's material existence.

Let's try to explore further the moment of the "image of the red book" and the attention that brought it to the surface. Merleau-Ponty describes attention as "still empty but already determinate intention" (2002, p. 33); in its central role it is *called forth* (ibid, p. 32) or *awakened* (p. 31) and brings about *"a passage from indistinctness to clarity"* (p. 32). The clarity here is the *identification* that the book we're looking for has such and such a colour, shape, size etc., as the image that surfaced in our consciousness. We need not be naïve about the form of the image that occurred, trapped by our *natural attitude* "of unquestioned belief in the world" (Husserl dictionary, p. 49): if we were looking for a *digital* form of the book, stored in our computer, without being able to recall the folder in which we had saved it, our attention would have been drawn towards other images, such as the title of the file, a distinctive phrase that only that book contained, or similar *profiles* (in Husserl's terms) of the book's *digital* form of existence, which would allow us to trace it, given the possibilities and limitations of the computer's search software capabilities. In other words, even the first object, i.e. the mental object of the "image of the red book" is far from being straightforward, as it is implied by Radford's example.

The sought book had a material form, and its existence as a *mental* object (the "image of the red book") started *after* our intention to look for it, as mentioned by Radford at the beginning of the citation: "we are standing in front of a wall covered by shelves full of books *without any particular intention to look at them*" (italics added). In order to understand the determinacy that our first attention

brought "[w]e must recognize the indeterminate as a positive phenomenon" (Merleau-Ponty, 2002, p. 7). From the state of "without any particular intention to look at" the books, we shifted to the state of a particular need, related to one of them; our *intention* to find the book was the condition for the possibility of the appearance of the *image* of the book. The intention is linked to a need, namely "to check something in" it. But the inaugurating moment of the mechanism of objectification from the viewpoint of my research was set at the *intentional moment,* when consciousness started looking for a particular object in order to fulfill it, and finding the image of the book, i.e. an *intended* mental object (i.e. in its *material* form), before we found the book itself.

A need emerges here, from the perspective of my Husserlian phenomenological investigation, namely to distinguish according to their directedness towards goals or ends,[43] the sign on the shelf and the "image of the red book". The sign on the shelf concerns the objectification of the book's *location,* and it is contrasted to the objectification of the *object sought* that is related to the "image of the red book", which constituted an "intended perception" (ibid., p. 39). The latter, i.e. the intended perception of the book*,* worked through the "image of the red book", which is a mental object that sprung from our "non-reflective perception" (ibid., p. 39). In scrutinizing the shelf the book was ultimately brought to *originary* sensuous perception—namely at our gaze and in our hands—and then to the *new* intention to facilitate our search the next time, expressed by the sign that we put on the shelf. The range of experience indicated by this contrast of goals that the two objectification moments bring to the surface, is quite symptomatic of the motivation(s) of my research: to lay bare the *intentionalities* of cognitive acts and to study "the act, the aim, the constituting mode of consciousness itself in the structuring of the *horizon"* (cf. next quotation) that *mathematical objectivities* and *mathematical objects* are rendered possible. This "double sense" of the "phenomena of knowledge" in the case of objectification is at the core of the Husserlian

[43] Their *teleology* (Moran & Cohen, 2012, p. 314).

phenomenological study of the modes of knowledge, as Husserl expresses it in the abstract of the 1907 Gottingen Lectures entitled *The Idea of Phenomenology*:

> the phenomenological study of the modes of knowledge must always be understood as the study of the *essence* from which is brought forth their inherent intentionality, that is, the possibility and the condition of the object of knowledge as well as of the knowledge of the object. Furthermore, this definition means that the phenomenology of *knowledge* is the science of the phenomena of knowledge in this double sense: on the one hand, science of knowledge as explication of appearances, figurations, presentations given to consciousness in which such *givens* configures themselves and become—either actively or passively—object *of* a consciousness *and,* on the other hand, analysis of this objectivity itself, that is of the act, the aim, the constituting mode of consciousness itself in the structuring of the *horizon* from which they are rendered possible. (Moran & Cohen*,* 2012, p. 253, italics in the original)

Objects for Husserl appear in consciousness and their objectivity is constituted through their appearance. Husserlian phenomenology as I apply it in my study is concerned with both the object as it is intended and constituted by consciousness, and its objectivity, i.e. the conditions and the possibilities of the appearance of the object to the fore of consciousness *as* being objective, i.e. as existing or potentially existing in the "world for us all". Objectivity is constituted through the object by the subject for Husserlian phenomenology, and in this sense objectivity is a *becoming,* involving two poles—the subject and the object—(Kolen, 2005), based on *immediate experience*. By drawing our attention to the "image of a small red book" and its appearance in our consciousness, before any sign was set, we are concerned not only with the object itself but also with "the possibility and the condition of the object of knowledge"; by trying to get closer to the object's origins we are engaging in the *phenomenological* "science of knowledge as explication of appearances"; by analysing this primal objectification, we cast light on these "appearances, figurations, presentations given to consciousness … [that] become—either actively or passively—object *of* a consciousness".

The focus of my analysis will be on objects that become *mathematical* objects as they are intended for this purpose, and as they are finally drawn out from the learner's lived experience, through *intuition*, rather than as objects deriving from mere attention. Deriving from *pregnant* intentionalities, as Husserl would call them, echoing Socrates and his method. Radford's "red book" example served the purpose of an introduction to the complexity and richness of the objectification process. The next section will use the same example in order to further adumbrate the objectification process and the longitude of the phenomenon from the "unreflected" to the different reflective layers of the final reflective experience of the "image of the small red book". In other words, the inquiry here will be from an archeological perspective (horizons), first theoretically, and then methodologically and analytically, in the following chapters.

1.3.4 Introduction to Husserl's intentionality, founded and founding acts, objective sense and determination of an object

In the example mentioned by Radford the intentions seem to be clear and straightforward, related to a particular empirical object, namely the book of Aristotle's *Poetics,* and the needs to *find* it and to *mark its position* on the shelf. Two different intentions arose due to two different tasks, and they gave rise to two different objects, i.e. the "image of a small red book" and the "mark or the sign" that concretised the objectification of the location of the book. The intention stated clearly by Radford is the one related to the "mark, or sign on the shelf", which is straightforwardly motivated by the task of locating the book more easily the next time. Radford's particular interest in the *semiotic means of objectification* (the phrase appears 33 times in his article) deliberately neglects to focus on the *mental* object that appeared due to the original need to *find* the book; the image that gave "rise to an intended perception" was put aside in favour of the objectification related to "the mark, the sign", and it was classified as the outcome of a "non-reflective perception".

Radford mentions that it was "non-reflective perception" that gave rise to an "intended perception" through the "image of a small red book", thus implying that

there was previous engagement with the actual book, and hence it was an act of recollection that allowed the image of the book to surface. Since my study is mainly concerned with *pre-semiotic means of objectification,* the following questions arise from my point of view:

- What knowledge can we gain from the "image of a small red book"? In which way is it an outcome of a "non-reflective perception"?
- How was the recollection (through which the "image of a small red book" surfaced) structured? How simple is the experience of the "image of a small red book"?
- What can the experience of "non-reflective perception" tell us about objectification, from a first-person perspective (as it is perceived by the subject)?

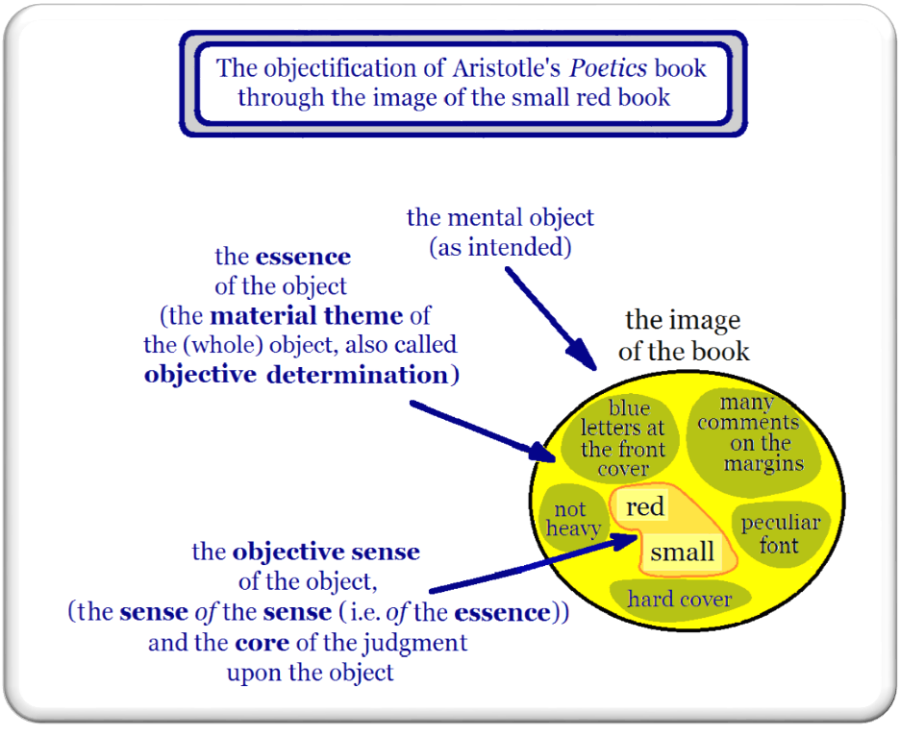
We know that the "image of a small red book" is not an original perception of the book, that it was not a *direct* outcome of pre-reflective activity, therefore it was *already mediated.* Moreover, the image was thematised as "red" and "small", which is another evidence that the reflective process had taken place. Although there is no doubt that one or more pre-reflective perceptions of the book were originally involved as *founding* acts (cf. Drummond, 2003 in *The New Husserl,* §6) to the final appearance of the image in consciousness, my claim, which will be scrutinized below, is that the image has its *origins* in pre-reflective activity (in *sensuous* experiences with the book), but it had already undergone a reflective process, even *before* it appeared in consciousness for the specific task.

In Husserl's words, the particular image is "experienceable only insofar as it is founded on what is simply and sensuously experienceable and only insofar as it is given in unity with what is simply 'there' according to perception" (Husserl, 1973, §12, p. 55). The image "is given in unity with what is simply 'there' according to perception", and Husserl's phrase echoes the previous encounters that 'we' had with the actual book, since "we suddenly remembered that we needed to check something in Aristotle's *Poetics*" (Radford, 2003, p. 39). But in order to understand the objectification of the "image of a small red book" due to the recollection that brought it to the surface of consciousness, we have to fully realise the difference

between the book itself and the image of the book, the object and the intentionally perceived object, i.e. the intentional object:

> The *tree pure and simple,* the physical thing belonging to nature, is anything but [*nichts weniger*] this *perceived tree as perceived* which, as perceptual sense, inseparably belongs to the perception. The tree simpliciter can burn up, be resolved into its chemical elements, etc. But the sense—the sense of *this* perception, something belonging necessarily to its essence—cannot burn up; it has no chemical elements, no forces, no real [*realen*] properties. (1983a, p. 216)

In the same sense, the "image of a small red book" is not a fiction but it is not the actual book either; it is the *intentional correlate* of the perception of the book from previous encounters with the real object. And the book as it is perceived has a *sense* according to Husserl, "something belonging necessarily to its *essence*". What does Husserl mean by "essence" of the image and which is the *sense* of the image that *necessarily belongs to its essence?* I will try to explain this with reference to the following diagram:



Graph 1. The objectification of Aristotle's *Poetics* book through the image of the small red book

Taking a closer look at the reflective process we notice that it is even more compound, as the example[44] of finding the *digital* version of Aristotle's *Poetics* indicates: would an "image of a small red book" appear if we were looking for the book in the files of our computer? Or, if it would appear, wouldn't we reject or modify it in favour of the title of the *file*, the *directory* in which we have stored it, or a *phrase* from the book that would render it accessible due to the particular software capabilities that we had available? This view of the reflective process brings to the fore the particular *materiality* of the image and the implicit *judgment* that valued the image as applicable to the particular task. Due to this judgment, our attention was led to "focus on some red books and, in practical terms… almost ignore the others" "[i]n scrutinizing the shelf", and in this sense, the judgment could provide the key for the "essence" and the "sense" of the "image of the small red book"'. For the Husserlian perspective adopted by my study,

[44] See section 1.4.1.

> [t]o judge is to grasp intentionally a state of affairs. … The object about which we judge is already given, for example, in a perception or remembrance, and in judging we distinguish features or parts or relations belonging to the object and make them the object of a special regard. … The act of judging, therefore, is directed toward the object so as to grasp it with respect to some categorial form or other. These categorial forms are, consequently, objective or ontological forms mirrored by the logical forms belonging to the senses in and through which we intend the categorically articulated objectivities. (Drummond in *The New Husserl,* 2003, pp. 83, 84)

Husserl explicates the act of judging that took place as directedness toward the "image of the red, small book" (the intended object) and as an *intentional* grasping of the image's "categorically articulated objectivities" of *redness* and *smallness* (the object as intended); with the latter being objective forms intended "in and through the senses". In other words, the judgment is a judgment upon a *mental object* (the "image of a small red book") but what is most important is that the act of judging distinguishes *"features…* belonging to the object and make[s] them the object of a special regard". These features, namely the qualities of "red" and "small", are the *objective sense(s)* corresponding to that object (cf. Graph 1). As Husserl explains, the *objective sense(s)* are "categorial forms" intended through the material texture of the image (the *theme* of the object), which is what he calls the *determination of the object:*

> The *objective sense* corresponding to such an object is, consequently, a *sense of* sense, a second-level sense. Therefore, *from* sense *as objective sense* we must distinguish *sense as the determination of an object.* Sense as objective determination belongs to the object itself as a theme. (Husserl, 1973, p. 269, emphasis in the original)

The materiality of the image (the expression of the book's material existence) is already a *sense* of the image, being the *theme* of the object (or *objective determination*) as Husserl calls it. And the qualities of "red" and "small" are a *sense*

of sense, a second-level sense, which are intended through the object's material sense in order to determine the action that the judgment entails.

The understanding of the judgment of the "image of the red, small book" as a suitable recollection for the action demanded by the task, goes through the thorough understanding of the *intentionality of the act,* i.e. the "directed[ness] toward the object so as to grasp it with respect to some categorial form or other" (Drummond, op. cit.). The material features of redness and smallness (the *objective sense(s)*) were already intended as senses of the book's material existence (the *determination of an object*) in previously re-presented[45] experiences, "in and through the senses" (objectively). More precisely, from the unthematising pre-reflective perception that redness and smallness originally occurred, or even thematised for reasons alien to the current situation, they were reflectively thematised as significant features, according to intentionalities and horizons of previous appearances. And due to the task of finding the book our *intentional* consciousness, our embodied *torchlight* cast a new light on such embodied experiences, as it was intending for the book's *particular* material senses (such as redness and smallness), through the *material theme* of the sought object. Thus, 'redness' and 'smallness' emerged as the objective sense(s) of the mental object's material existence, as the core or part of the core of the mental object's material determination; henceforth they became the core of the judgment upon representing the object itself as appropriate, and they manifested their significance in the action that followed, the search for small, red books in the library. The description of the experience as a whole echoes Husserl's words concerning the embodiment of the essence (the material theme) of the mental object and the constitution of significance (through the judgment): "It belongs to the essence of objectivities *of* sense not to be otherwise than in real embodiments whose significance they constitute" (Husserl, 1973, p. 268).

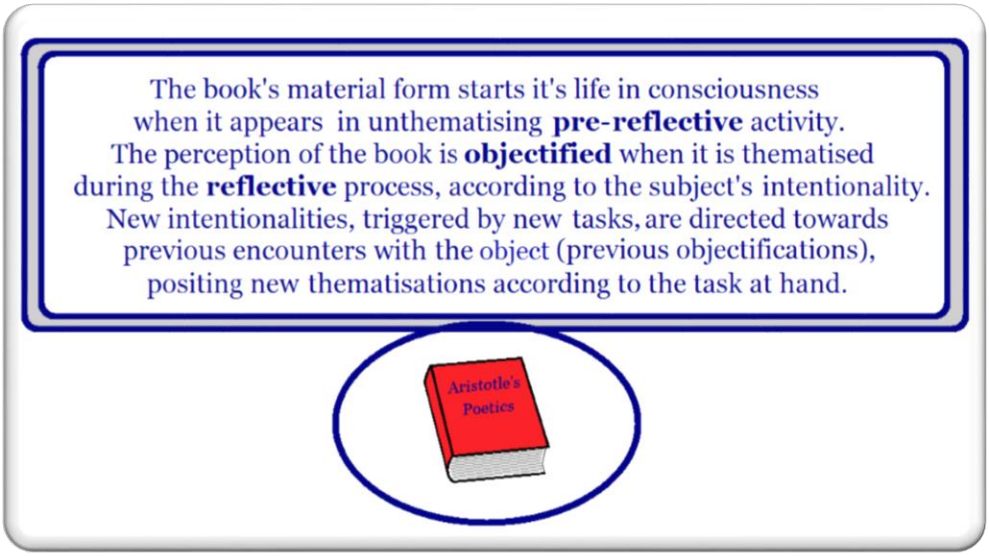
[45] For an explication of re-presenting (*Vergegenwärtigung*) or making–present in Husserl, see Drummond, p. 78 (in *The New Husserl*, 2003).

In summary, I refer back to the four questions that I enunciated earlier. Firstly, from the analysis of the objectification of the "image of a small red book" it became apparent that it is much more than the outcome of a "non-reflective perception". Secondly, the experience that brought the image to the surface of consciousness was shown to be far from simple, and the value of the Husserlian phenomenological analysis was manifested in allowing us to take a deep gaze at the apparatus of the image's appearance;; starting from *founding* pre-reflective appearance of the book images, to *founded* thematised objectivities of previous *reflective* encounters with the book, up to the particular intentionalities generated by the task of finding the book, and finally to the judgment that the image is appropriate and the particular actions that followed. Thirdly, since the image of the book was already mediated due to previous experiences with the book (through the recollection that brought it to the surface), I do not analyse here the structure and the possibilities of appearance of the pre-reflective experience(s), as well as what the latter can tell us about objectification. This third question is addressed later both empirically and theoretically. What is also stressed is the links between sensuous perceptions such as the "small, red book" example, and abstract mathematical objectivities.

The image of the red book served as an example in order to explore the essential links between *objectification* and *meaning* or, more precisely, *objectification* and *sense.*[46] In order to unfold the Husserlian approach of my study on this crucial topic I investigated the "concrete-object-in-the-How-of-its-intentional-appearance" (Held, in Welton, 2003, p. 40). Analytical tools emerged around the concept of the object, such as *intentionality, fulfillment* (of the intentionality), judgment, *founding* and *founded* acts, *essence* (or *theme* or *objective determination*), *objective sense* (or sense *of* sense). The object appeared

[46] The double connotation of the word sense is particularly expressed in its use as a verb, e.g. "I sense", "I make sense", (a) one of them being "to perceive (a sensation, object, etc.) by a sense or senses" and (b) the other one to "understand, comprehend, grasp, (meaning, import, etc.)" (Shorter Oxford English Dictionary, p. 2754); and it is a 'gap' (tension) that is implicitly explored in this study. For the Husserlian distinction between *sense* and *meaning* cf. *Index/Glossary* for *sense*.

in consciousness as something "throw[n]… in front of somebody" (Radford, 2003, p. 40),[47] in a process where the intentionality of consciousness was the driving force and the constituting agent. As we shall see in a following section, the constitution of an object (i.e. objectification) starts from such moments according to Husserl, although it is intuition rather than attention that is the decisive act that brings the object to the surface of the embodied consciousness, of the *lived-body*.



Graph 2. Summary of the objectification of the "small red book".

1.3.5 Perception and remembering – Clarifications concerning the *transcendental reduction* that was practiced in the previous section

In the previous section I analysed the objectification of the "image of the red book", which was a product of remembering; the analytical tools such as *intentionality* and its fulfillment, the *determination of the object* and the *objective sense* were introduced as key factors in understanding how an object appears in consciousness. But it is vital to clarify that I am following a *phenomenological* approach rather than

[47] For the etymology of objectification cf. Radford, 2003, p. 40.

a psychologistic[48] one, since the misunderstanding concerning the latter would always entail the danger that my analysis can be interpreted to mean:

- The study of how people actually think or ought to think, and
- The collapsing of the logical nature of the judgment [49] into private psychological acts.[50]
- The "highly plausible conception" of *logical psychologism,* that equates "*the formations produced by judging* and then, naturally, of all similar formations produced by rational acts of any other sort) *with phenomena appearing in internal experience"* (Husserl, 1969, p. 154); in this sense "concepts, judgments, arguments, proofs, theories, would be psychic occurrences; and logic would be, as John Stuart Mill said it is, a 'part, or branch, of psychology'" (Husserl, Formal and Transcendental Logic, p. 154)

The departure of my reflection was that "we have previously seen and held this book in our hands", and in this sense my starting point was completely unphenomenological; but I carried out the *transcendental reduction* on it, by bracketing the spatio-temporal mundane actuality of book and author and referring to the previous reflective and pre-reflective ("unreflected" in Radford's terms) experiences with the book "as present facts" and "according to their remembered intentional content" (Husserl, 2001, p. 453). Thus I gained, as Husserl puts it (ibid.), the transcendental phenomenon, namely, the past transcendental phenomenon of the ego's earlier perception of the book, whereby the real book itself was only the *intentional object* of the seeing and the touching acts. The key for the reduction on the experience of "the small red book" was the inherent [51] directedness of consciousness (its *intentionality*), and my drive was towards unveiling the particular apparatus of its fulfillment, which is a purely *phenomenological* method. By

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[48] For a short historical account of psychologism, and the basic arguments concerning the controversy that it created see the *Glossary – Index of terms* (at the end of the Thesis).

[49] E.g. the judgment that the "image of the red book" was an appropriate mental object for the corresponding task of finding the actual book.

[50] Similar to the case between Husserl and Frege, after the former released his first book (cf. Moran & Cohen, 2012, p. 273).

[51] *'Immanent'* in Husserl's terms.

implementing such an approach I avoided examining the laws of thought that govern the phenomenon as being related to empirical psychological laws, as *psychologism* would have it. And in Husserl's own words, the transcendental phenomenological enquiry that was adopted here can be effective in studying the whole range of recollective experiences:

> In this way I can gain the entire realm of my memories as phenomenologically reduced, thus the realm of all memories of objectively mundane things and processes, and then of all memories in general, e.g., memories of mathematical proofs that I have carried out; and I gain them not only as present facts, but according to their remembered intentional content. And what finally results from this is my (i.e., the Ego's) past transcendental life with all of its past poles which, insofar as they are object poles, are bracketed, while the ego that is everywhere identical is the transcendental ego, and should not be bracketed, just as little as the past transcendental lived-experiences. (ibid., p. 453)

By presenting the recollection of the red book as *"present fact"* and *"according to [its] remembered intentional content"* I brought the "image of the red book" under the Husserlian phenomenological lens. I bracketed "all of its past poles… insofar as they are object poles", namely the previous reflective experiences with the actual book, by considering them as merely thematising the book's "features or parts or relations belonging to the object" (Drummond, 2003, op. cit.), according to the previous intentionalities, to which I have had neither access nor particular interest for the purpose of the transcendental reduction. But,

> I only put out of play the positing, the memorial belief in the past objectivity, but not the belief implied in it, namely, the belief in my past ego and my past lived-experiencing, and my past perceiving in which my past life was given perceptually. (Husserl, 2001, §368, p. 454)

And I maintained the *transcendental* feature of the ego as ever producing, as ever living through new *thematisations* of the book's features, through new involvements with it, through new appearances in the *embodied consciousness,* such as the image

produced by the 'current' task of finding the Aristotle's *Poetics* book. I went through such a process since the *embodied consciousness* and the body as *subject—which is the same thing* according to Husserl and Merleau-Ponty—are 'equally' transcendental to the transcendental objects that 'they' encounter, such as the concrete "small red book". The embodied consciousness and the book are 'equally' transcendental in the sense that the lived body (i.e. the body as subject) actively and potentially enacts uncountable perspectives and intentionalities towards the book, as the different 'profiles' that the book may reveal to the subject. [52]

Thus I arrived at the qualities of the book (red, small), which were targeted and became *evidence* for and due to the phenomenological perspective of my study, as previously spotted (thematised) and 'currently' *separated* from the material *theme* of the image (its *essence*) by the involved subject.[53] I followed "the *evidence* that itself lies, in part, in the phenomenological reflection on the present, in part, in the phenomenological reflection on the past (that is, in the reflection penetrating into the *intentional content* of memories)" (ibid., emphasis added),[54] without asking "whether or not this evidence is apodictic, whether or not it is better than the evidence, e.g., of external perception" (ibid.).

In summary,

- The structure through which this evidence came to the fore included:
  - ø The *phenomenological analytical tools* of
    - *intentionality,* (§§ 1.4.4, 1.4.5, 1.4.6, 1.5.1, 1.5.2)
    - *fulfillment* (of the intentionality), (§§ 1.3.1, 1.3.3, 1.4.4, 1.4.5, 1.5.2)

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[52] C.f. "Husserl insists that there is no apodictic final evidence about empirical entities, due to their perspectival mode of givenness" (Moran & Cohen, 2012, p. 114); also cf. Husserl, 1999a, §34, and Husserl's concept of "the de facto transcendental ego" in p. 70, p. 71.

[53] Even if the 'involved subject' my approach involves a process that Husserl would call *imaginative free variation* (cf. Husserl's example in 1999a, § 34, p. 70)

[54] "Husserl often speaks indifferently of *phenomenological* and *transcendental* reductions or indeed of the 'transcendental-phenomenological reduction' (CM § 8, p. 21; Hua I 61; Crisis VI 239)" (Moran & Cohen, 2012, p. 274), and this is the way that I use it in this case. But I attempted to exemplify in this section the particular *transcendental* character of the reduction.

- the *implicit judgment* and the *constitution of significance* (through the judgment),
- *founding* and *founded* acts, (§ 1.3.4)
- *essence* (or *objective determination* or *theme* of the object),
- *objective sense* (or sense *of* sense), (§§ 1.3.4, 1.4.2, 1.5.2) and

ø The *phenomenological* perception of

- The configurations involved in the 'red and small book' example, such as:
  - the act of recollection,
  - the non-thematising pre-reflective perception(s)
  - the transition through immediate experiences to the reflective perceptions in previous appearances/objectivations, and
  - the new thematisations
- The *contents* found in the *description of the example,* which fulfilled the structures applied by the phenomenological perception, e.g.
  - the *thematised qualities* of red and small,
  - the *transition* from the lack of attention to the emergence of attention,
  - the particular *implicit judgment* and its *filling* by the qualities of 'red' and 'small'.

- My concern was to release *a new layer of meaning* through the phenomenological investigation of the 'red book' experience, which was
  - ø *limited* by its fictional character and the consequent lack of any account of the immediate firsthand experience(s),
  - ø *achieving a generality* through its common features with conscious acts that I, as every reader of this study may recognise, and
  - ø offering a different perspective than Radford's, namely a *pre-semiotic* approach to objectification, that sees "*object* and *something real,*

actuality and real actuality … sharply separated" [55] and objects "defined as anything at all, e.g., as subject[s] of a true (categorial, affirmative) statement", "ascrib[ing] to them, as to other objects, actual (veritable) being" (1983a, p. 41; also cf. Tragesser, 1977, e.g. p. 35).

Thus the analysis of the aforementioned episode became an example of Husserl's *transcendental reduction* in the case of rememberings, as I aimed at infiltrating into the *intentive mental processes* and their intentional contents (1983a, §36).

[55] In a later section I will introduce the essential aspect of Husserl's approach to mental objects, i.e. to "Husserl's distinctive view … that beings can actually be given to us, that 'to be capable of being given belongs to the essence of being'" (Hintikka, 1995, p. 84). The basic idea is that "[w]hat is immediately given to me will then at the same time be part of the mind-independent reality and an element of my consciousness. There has to be an actual interface or overlap of my consciousness and reality. …reality in fact impinges directly on my consciousness" (ibid. pp. 82, 83).

CHAPTER 4. The living body, *operative* intentionality and intentionality of act – The milieu of objectification and intuitions

1.4.1 Transcendence of objects and new knowledge

When we see the side of a spatial object we do not take the side as the object. "It is clear that a non-intuitive pointing beyond or indicating is what characterizes the side actually seen as a mere side, and what provides for the fact that the side is not taken for the thing" (Husserl, 2001, p 41). This phenomenon, neglected in our *natural attitude* of the world as simply existing, becomes evident when the world is approached as a phenomenon, when we try to understand the *how* of the world's appearance and constitution by the subject as *always there* (Husserl, 1969, p. 225; 1970a, p. 123), as "the world for us all" (Husserl, 1969, p. 236) and as *transcending* our understandings of it. "Transcendent" for Husserl means "inexhaustible" (Follesdal, in Dreyfous & Wrathhal, 2006, p. 109) and as we shall argue, this is true not only for the inexhaustible *sides of the object* | *profiles of the real* that the cognising subject may come to contact with, but it is also true for the *cognising subject* that constitutes the object, and testifies her consciousness as an objectifying consciousness, while producing a first sense of objectivity.

1.4.2 Empty indicating, referring to possible new perceptions, and what is originally exhibited. Synthesising identifications anew, through the synthesis of coincidence

Husserl's idea is simple, although groundbreaking in its relation to the acquisition of knowledge: for Husserl, "something transcending the side is intended in consciousness *as perceived"* (Husserl, 2001, p. 41, my emphasis). We see that Husserl is shifting from the object-in-itself (as Kant would have it) to the object as it appears to consciousness, "as perceived"; while the actual perception for Husserl is a mixture of "actual exhibiting", which presents "in an intuitive manner what is originally exhibited", and "of an empty indicating that refers to possible new perceptions" (ibid., p. 41).

Thus the knowledge production has for Husserl a double nature, expressed by the *immediacy* of intuition and the *anticipatory* "empty indicating that refers to possible new perceptions". This "empty indicating" will be filled intuitively, one way or the other (Husserl, 2001, p. 378), resulting to a *closer determination* or to the *collapse* of the *previous* and the *just-present* objective sense of the object involved:

The perception of something *transcendent*—the perception of the beer bottle— can 'deceive.' That its object exists, and exists in this way, can be 'doubted.' Perhaps the beer bottle is 'other than it appears'; perhaps it does not exist 'at all.' What does this mean? It means that the perception includes intentions that are still unfulfilled and that it can either undergo fulfillment or 'closer determination', or 'contradiction' and 'disappointment.' (Husserl, 1991, p. 246)

The "empty indicating" allows itself the freedom to be wrong or just right for another activity, for another approach of the same, for a related or for a so-far-unrelated object, and for the different set of intuitive horizons that goes with it.

This "empty indicating" is nevertheless the key to *"possible new perceptions",* and this is a focal point of my research, since I perceive the *renewal* of knowledge as inevitably starting intrapsychically, from "new perceptions" and their immediate application and absorbedness in epistemic-social-cultural-historical spatiotemporal instantiations. The fact that new concepts end up as theorems in textbooks does not alter their origins in *"possible new perceptions",* which were once brought to the fore of consciousness. The fact that some of these new concepts *survive* is a testification of the *constitutive effect* of the cognising subject (the *ego pole*) in the first place, which is a pole that comes to terms with the *object pole,* through the *intentional* subjective act of objectification (cf. Husserl, 1970a, pp. 171-172).

The question posed for the first time precisely by Descartes, namely "how is it possible, for immanent knowledge, to grasp something that is not immanent" (Vanzago, 2005),[56] is addressed by Husserl through his conception of *immanent*

[56] See Part 1, Ch.2 for the theoretical discussion of the same question, in the context of the distinction between Husserl's and Kant's approaches to experience. Immanent transcendence is also crucial for the Husserlian response to Meno's paradox.

transcendence, which is *achieved* by the transcendental ego.[57] The "empty indicating" is meaningful due to the object's *transcendence* and the *awareness* of this transcendence by the subject, and the latter is possible because of the ego's *immanent transcendence*—i.e. the ego's survival as an identical ego despite its diverse manifestations, and the potential and actual inexhaustible ways to perceive objects. This interplay of *transcendence* on behalf of the object and *immanent transcendence* on behalf of the subject is another Husserlian innovation, which is responsible for the constitution of sense:

> Transcendence in every form is an immanent existential characteristic, constituted within the ego. Every imaginable sense, every imaginable being, whether the latter is called immanent or transcendent, falls within the domain of transcendental subjectivity, as the subjectivity that constitutes sense and being. The attempt to conceive the universe of true being as something lying outside the universe of possible consciousness, possible knowledge, possible evidence, the two being related to one other merely externally by a rigid low, is nonsensical. (Husserl, 1999, p. 85)

Objectification is for consciousness an *existential* act, since "consciousness is nothing without its object, for the intentional definition of consciousness is in terms of *consciousness-of"* (Vanzago, 2005, p. 360). And we could perhaps paraphrase Descartes in saying that for Husserl consciousness is best_characterised by the phrase "I objectify therefore I am". But Husserl's answer does not consist, as in Descartes, in a deduction or a demonstration of the existence of transcendence, but in a pure clarification of the essential possibilities of consciousness, which brings the structures of 'pure' consciousness itself to light through a "complex concept" that Husserl discerned as one of its main features:

> Intentionality is a complex concept: it means that the world is to be seen as 'world-for-consciousness', [as] the object of possible experience. But it also means that consciousness is nothing without its object, for the intentional

[57] These terms are only introduced here and they will be explored in the following sections.

> definition of consciousness is in terms of *consciousness-of.* The achievement of the reduction is then the awareness that consciousness and world are to be seen as poles of a relation which, as such, becomes the real theme of phenomenology. (p. 360)

And in this sense "the real theme" of my study is the "awareness that consciousness and world are to be seen as poles of a relation" which yields objects on the ground of lived experience. The empty indicating may be filled or unfilled, and may conflict with the raw experience:

> [W]hen we perceive, some of the 'anticipations' in our noema are filled by hyle, others are not; they just point to further features of the object and may become filled when we go on exploring the object. These unfilled anticipations may conflict with the hyletic experiences we get when we explore the object, in that case, an "explosion" of the noema takes place, we have to revise our conception of what we perceive, we have to come up with another noema that fits in with our hyletic experiences. The hyle therefore *constrain* the noesis we can have in a given situation and thereby what noema we can have. (Follesdal in Dreyfus, 2006, p. 108)

Although the constraints put by the reality, which appear in the form of raw empirical material, the *hyle*, are precisely/well described in the aforementioned extract, there is something missed by Follesdal, since his reduction brackets the *perceived* object *as perceived*, by isolating what is intended in the object, *as intended*, i.e. by isolating the *noematic core*[58] from the object and the positing qualities. What is missed is the *object as perceived*, the awareness of the inexhaustibility of the object, as perceived by the subject, which does not stop us from perceiving the object always as a whole, [59] even when we don't yet have a name for it. It is what Drummond calls the *full noema,* i.e. "the union of noematic

[58] Which is the core of the *noema*, of the object as it is intended, also called the *objective sense* (Husserl, 1973, p. 269 – cf. the "red book example" and Graph 1 in Part 1, §1.3.4)

[59] "But what has been constituted as present in the Now is an absolute self that does not have any unfamiliar sides" (Husserl, 2001, p. 59).

sense and thetic character" (Drummond in the New Husserl, 2003, p. 72). We perceive the object always as a whole through our body-subject (*Leib*), with which "we can, whenever we like, orchestrate an intuitive presentational course, a reproductive course of aspects through which the non-visible side of the thing would be presented to us" (Husserl, 2001, pp. 40-41).

The object is prepared in the pre-reflective *passive* stage of our lived-experience, taking the form of object-like multiplicities of the *hyle*,[60] from where it is drawn with its *objective sense*,[61] although this sense may change, and/or it may change names in the course of the experience. The intentionality of *protention*, of anticipatory expectations that perceive the object-like entity as a whole, even before it becomes an object, goes side-by-side with the intuitive primal impression, i.e. the actual Now-moment of some *thing* that is originally exhibited.

The subject is in-the-world, in Merleau-Ponty's bridging of Heidegger and Husserl. The question is, as Husserl put it, the "distinction between determinations with respect to the content of the object [a] that are actually there, appearing *in the flesh*, and [b] those that are still ambiguously prefigured in full emptiness" (ibid, p. 43). Intuition is *in the flesh* of the lived-experience, as it makes possible the grounding of knowledge as a non-arbitrary process, based on actual/factual lived-experience. Moreover, intuitions make possible the presentifications of protentive prefigurations of the same object. As Husserl put it, "what is perceived is given in adumbrations in such a way that the particular givenness refers to something else that is not-given, as what is not given belonging to the same object" (ibid. p. 41).

This is true, one might object, only for the concrete empirical objects, which remain (relatively) unaltered through the course of our contact with them. What would guarantee the same or anywhere close to the transcendence of empirical

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[60] 'Hyle', 'hyletic' are Husserlian terms that will be explored throughout the study; see following section for a closer determination of hyle, which is not mere sense-data (also cf. the *Glossary – Index of terms* at the end of the Thesis).

[61] 'Objective sense' or 'noematic sense' is another Husserlian term that will be explored throughout the study. It is meant here quite literally, as the sense that we endow to an object, such as a table, a house, a bird singing, etc.

objects, of the transcendence of abstract objects,[62] since the latter are constituted in *immanent* rather than *transcendental* manners of giveness? Husserl's position concerned the meeting of the ideal and the concrete, since anything ideal has a time-specific genesis in space and time (Husserl, 1970a), in Husserl's phenomenological attitude. In other words, Husserl's task was to *reverse* the Cartesian analytical method by starting from the bottom, from the lived-experience and the a priori of the Life-world, up to the (constituted as) ideal forms. Moving from the intentions and how they are met (fulfilled), to the object pole, as it is constituted subjectively by the ego pole. Husserl introduces the fundamental demand of *apodicticity,* (e.g. Husserl, 1970a, pp. 339-340), which is safeguarded by *intuition* and it rules out *psychologism* (Held, in Welton, 2003). Husserl's interplay between reality as self-given and subjectivity that is always intentional and always oriented towards the object-pole, this tension between the *ideal* forms (the 5th stage of objectification in the *Origins*) and their genesis in the Life-world, through the *living body*, just before they'll enter the intersubjective circulation, is a tension that Husserlian phenomenology brought to the surface of investigation, and one that my research takes position in favour of the *lived experience* of *mathematics-in-the-making*. Examples of such *lived-experiences* are analysed in all three cases of the prospective teachers of mathematics, attempting to exemplify that such moments are far from rare, that our "new-found Land" (John Donn) was always here, only that without the phenomenological analytical instruments such moments are inaccessible, invisible, silenced. The 'glasses' bring the object to a new existence, where interesting details of the cognitive structure are lit, due to the new perspective.

The discovery of this tension is clearly stated in Husserl (1970a, p. 340), as "the discovery of the radical contrast between what is usually called apodictic knowledge and what, in the transcendental understanding, outlines the primal ground and the primal method of all philosophy". If the calling is for a

[62] *Mathematical* is another name for what has been called *abstract* throughout this study, which is concerned with the transition from the pre-mathematical to the mathematical knowledge constitution.

phenomenological attitude, then, as Husserl put it, it concerns "[man's] existence (sic) in the spatiotemporally pregiven world as the self-objectification of transcendental subjectivity and its being, its constituting life" (ibid.). In a deeper layer this research aims at contributing to this objectification, through a close view of the learners' constitutive activities.

As Husserl put it in his *Idea of Phenomenology* (p. 62), "the intuitive, direct apprehension and possession of the cogitatio[63] is itself a knowing, the *cogitationes*[64] are the first absolute items of givenness". Taking into account Husserl's attempt to ground his epistemology on the strictest possible account of *true philosophy*, we notice his recourse to his motor drive, *intuition*, "the intuitive, direct apprehension and possession of the cogitatio". Only that Husserl troubles these acts of thought that aim at the cogitatio, attempting to delve deeper than Kant and Descartes in object constitution, towards the objects' genetic, temporal, pre-structural life in passive synthesis, which inspired Merleau-Ponty's primacy of perception (Reuter, 1999).

The link between the *active* object constitution and its *passive* grounding is for Husserl and for my study *operative intentionality,* and its crucial[65] preparation of the object that will emerge from its object-like state. Operative intentionality is responsible for the *embodied* certainty of the learner's profile of her/his world-as-lived, which is at play every moment, temporalised every moment. Operative intentionality orchestrates the embodied experience as an expression of the lived-body—which is what Husserl terms as the *empirical ego*—in the passive (pre-reflective) stage of lived-experience. This phenomenological perspective of (cognitive lived experience was introduced by Husserl but focused on and radically advanced by Merleau-Ponty.

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[63] Acts of thought.

[64] The objects of thought in which the acts of thought are directed.

[65] Crucial for this research's effort to explicate objectivation, that is objectification's earliest stages (see below in §1.4.3).

1.4.3 Transcendence of objects, lived-experience and …'turning coffee into theorems' – An introduction to the three cases of the study

In order to understand the constitution of abstract mathematical objects and their structural as well as factual affiliations with the empirical ones, the phenomenological attitude towards the transcendence of spatial objects is already informative of the acquisition of knowledge: I walk around the house and I realise that it is a scenery, moreover, a scenery without interior.

The first realisation makes my previous perception of the house to collapse—to "explode", in Husserl's terms—while the second one is merely a specification of my new perception of the house, since I already anticipate that a scenery house might not have any interior. The sequence *house – scenery – scenery-without-interior* is not brought up here as an uncommon one, but rather as representing a thriving category:

- I talk with my friend on Skype, there is light filling her room, I suddenly hear a bird singing and I conclude: "it must be a nice day there my dear, and the birds are singing". She replies that "the day is rather nice though cloudy but this was my '12 o' clock bird', since my clock here produces different bird songs for different hours". The bird singing that misled me, either a representation of a real bird or not, affected as it was by the quality of the Skype connection, is another collapse of the object of my consciousness (the *noema*), and it is responsible for a change of perception due to the conflict, in order to coincide with the hyletic material of the experience. Now that I know that it was not an actual bird singing that moment, in my friend's garden, the apprehension of the bird that I previously heard singing becomes clearer than ever, finally settling as a "12 o' clock bird, at my friend's home, which I erroneously perceived as an actual bird singing". The previous perception does not vanish but remains as a cancelled one.
- A student of mathematics realises that an asymmetrical pattern gives better results and in a consistent way for the distribution in space, of "cops able to see two blocks away"; the asymmetrical distribution allows him to fully

mathematise his lived experience. A collapse of his previous conviction about *symmetrical* structures was *necessary* in order to unlock the task.[66]

- A student of mathematics realises that the argumentation she employs for a task does not produce significant results in the case that she tries to apply it; but the ineffective argumentation may be transformed—due to its generalisation and its change of set of values (its actual re-presentation)—to an argumentation that produces significant results, for the same case, in a process where the previous argumentation, by using abstract, mathematical notification, may be retained as a cell, as a pre-structure, acquiring new context. What previously attained local, trivial results, now holds general depiction of a structure and, under the light of the new results, why is it that such a constitution is often dismissed as irrelevant to the tasks of the course, of any mathematics course?
- A student of mathematics attempts to embody and then formalise her lived experience, of what kind of curve is one that is equidistant from a wall and a fixed point; after finding 5 points she intuits the curve as not crossing the x-axis. But after implementing a newly acquired tool her intuitive conception of the curve collapses, and it becomes an x^2 curve, instead of a part of it.[67]

There is a thriving list of cases where the 'fractal balloon' one,[68] just mentioned, as well as the other two examples, are but mere examples of everyday practices that remain neglected, concealed or cancelled, as insignificant, or by aiming at the de-subjectification of the *pregnant* lived-experience, which is already an aim of objectivism, often employing dialectics in a materialist sense. A student or a research mathematician may build up on her mathematical proof interface, and different mathematical categories may become variables in her proof zone. And it can't be that rare, as far as examples like Paul Erdős are concerned, who coined the aphorism "A mathematician is a device turning coffee into theorems", if not

[66] Reference to Ivan's case study (Part 3, Ch. 2).
[67] Reference to Mary's case (Part 3, Ch. 4).
[68] See Diana's case (Part 3, Ch. 3) and the evolvement of her empirical material to a generalised method, as an example related to the last bullet.

described himself as such by Renyi (Hoffman, P., 1998), and insofar as a quarter of a million new theorems are proved every year (ibid., p. 204).[69]

1.4.4 Protentions and operative intentionality, immanent to the transcendental subject, and pointing towards a particular reduction, towards a phenomenological method.

According to Husserl it is the empty indications that realise new perceptions, as soon as the latter will fulfill the former intuitively. I see a man sitting solemnly in the dusk; after a while I realise that it was an illusion, due to a costume that was hanging on the chair, a shadow that was mistaken for the head and the poor light. My conscious self strives for closer determination, as it starts with the image of a "solemn man, sitting in the dusk". Any man or woman has a *living body,* as I have, something radically different compared to hats, chairs etc.. My perceptual "tendency toward originarity" and the intention to understand better what sort of living body is the one that seemingly appears in my sensuous field, leads to the collapse of the previous *noema*, i.e. the collapse of the *alien* subjectivity that was *already* understood as a whole, alien for my world, although different than any empirical object.[70] My *"purposeful striving" toward evidence* through *closer determination* is the implicit *intentionality* that challenged my previous perception and led to its collapse.

This is a view that I achieve when I apply the transcendental reduction, by excluding everything external to the object as it appears, thus arriving at the *genetic* settings of perception, and the *structures* that reign its inaugural moments, by *necessity* rather than *reason*. By exploring the *potentiabilities* (Held in Welton, 2003, pp. 14, 19) of conscious life, I'm getting closer to understanding the Husserlian necessity and the origination of knowledge.

> In order to be able to be consciousness-of-something at all, consciousness must know of its own potential (*Möglichkeit*)—Husserl coins a striking

[69] See Wikipedia, http://en.wikipedia.org/wiki/Theorem#cite_note-11.
[70] Such as the costume that misled me.

> conceptual term here, 'potentiability' (*Vermöglichkeit*)—to allow the related 'something' to appear in intuition. Consciousness's intending of an object is, therefore, not a static relation-to-something, but instead is animated with its tendency toward originarity; Husserl uses the words 'intention' and 'intending' throughout his works in a way similar to our daily usage, indicating a purposeful striving. Intentional consciousness is, in all its forms, focused on finding satisfaction in the intuited self-having of lived experience. Consciousness wants to go toward evidence; that is what forms its goal, its telos. In this sense, all conscious life—as Husserl would say in his later work—rests under the rule of a 'teleology.' (ibid., p. 14)

My *expectation* shifted from the perception of "a man sitting solemnly in the dusk" to a conglomeration of objects, which made me *think* that someone was there. In both cases there is nothing 'wrong' from the phenomenological point of view, since these are two perceptual acts (*noeses*) with their corresponding objects (*noemas*, in Husserl's terms). But since "I have the consciousness 'I can…'", since "it is within my full power, my 'capability,' to follow the referential interconnections in any direction I choose" (ibid., p. 19), and since my perceptual consciousness finds "satisfaction in the intuited self-having of lived experience", only one of these two apparitions may survive, the one that corresponds better to the intuited self-having of my lived experience. Another perspective within the phenomenological one that is adopted here concerns the shifting from the intentional horizons that perceived a certain sight as a "solemn man, sitting in the dusk", to different intentional horizons, always accompanied by *empty indications* (expectations), that strive for fulfilment. Such empty indications are a "halo of consciousness" for Husserl…

> It is an emptiness that is not a nothingness, but an emptiness to be filled-out; it is a *determinable inditerrminacy*. … In spite of its emptiness, the sense of this halo of consciousness is a *prefiguring* that prescribes a rule for the transition to new actualizing appearances. (Husserl, 2001, p. 42, our emphasis).

What makes all this experience possible is the pole of the subject, of the transcendental subject, responsible for the unification of the raw material of the simple acts (Husserl, 1973, Introduction, §§ 11-14), in the pre-reflective and the reflective realm of her lived-experience. The transcendental subject whose consciousness is always intentional, is always consciousness *of* something.

> But already within passivity we find an intention, as belonging to the structure of intentional lived-experiences as such *without the ego coming into play as a subject who is [actively] directed,* as a cognitively striving subject potentially making a decision. It is what in truth transforms a concrete lived-experience, perhaps like a remembering or an expectation already stirring in the background, into an intentional lived-experience. (Husserl, 2001, p. 449, my emphasis)

1.4.5 Operative intentionality and the living body, operating in the passive stage

It is crystal-clear in Husserl's epistemology that the real has a share in the transcendental subject's lived-experience, that there is determinable *indeterminacy*, and that "already within passivity we find an intention, as belonging to the structure of intentional lived experiences as such", "without *the ego coming into play as a subject who is [actively] directed,* as a cognitively striving subject potentially making a decision". The response of the (transcendental) ego to the real is *objectification,*[71] while the latter is grounded on the direct (intuitive) embodied contact with the real. The real is a primal agent in objectification, and intuition, in its primordial stage, grounds it as such. The adumbrations of the object, given in primal impressions, are *combined* with empty anticipations (protentive tendencies) that some of them will be fulfilled, when the determination of the object will be fuller, seen from different sides etc.

[71] Rather, *objectivation*, which is the startup of objectification. See later section for the elaboration of objectivation.

The operative, pre-reflective intention is immanent to consciousness, as "belonging to the structure of intentional lived-experiences", and it "transforms a concrete lived-experience, perhaps like a remembering or an expectation already stirring in the background, into an intentional lived-experience" (Husserl, 2001, p. 419). We are in the area of passivity (Husserl, 2001, p. 419)[72] and the primordial part that intentionality and intuition play in it. So let's take a look. Since there is no ego-cogito intervention the intention is of the type *"I can"* rather than *"I think"*. It is a pre-reflective intentionality and it is what Merleau-Ponty called *operative intentionality*, which is

> that which produces the natural and antepredicative unity of the world and of our life, being apparent in our desires, our evaluations and in the landscape we see, more clearly than in objective knowledge, and furnishing the text which our knowledge tries to translate into precise language. (Merleau-Ponty, 2002a, p. xx)

With the interplay of intuition and protentive tendencies, a unifying intention allows embodied intentional consciousness to distinguish objectlike entities from raw (hyletic) material and transform them into empirical objects. The objectilike entities reside consciousness until operative intentionality will call them forth to new thematisations, new unities that are constitutive for new objects.

Embodied consciousness is what Husserl and Merleau-Ponty call the *living body,* "which is never absent from the perceptual field" (Husserl, 1970a, p. 106, also cf. Merleau- Ponty, 2002a, p. 87, and Tito, 1990, p. 185). The body[73] plays an essential role in the fulfillment of *operative* intentionality, since it is "a sensory-motor behavior through which the world is constituted for man (sic) as the world of human consciousness" (Merleau-Ponty, 1964, p. xvi). The *operative intentionality* is "already at work before any positing or any judgment" (Merleau-Ponty, 2002a, p. 498) and it "produces the natural and antepredicative unity of the world and of our life" (Merleau-Ponty, 2002a, p. xx). It is pre-reflective but this does not mean that

[72] Also called passive stage (cf. Index/Glossary).

[73] The body as (intentional) subject, not as an object (cf. Reuter, 1999, pp. 71-72).

it is unconscious (Reuter, 1999, p. 76); and it can be revealed:

> We do not see the operation, since it operates. We see what would be missing without it; we circumscribe it as what makes speech be a "speaking of" ... and not be the conscious having of the idealities implied in speech. (Merleau- Ponty, 2002b, p. 44)[74]

When Mary, one of the three cases that are analysed in this study reported that she acquired a *bird's-eye-view* on her "little diagram", she quite literally expressed what 'bird's-eye-view' actually means and how the body plays on this transformation kinaesthetically, in sensori-motor co-ordination. It is a subtle but most crucial difference between Husserlian and Kantian epistemologies, that in Husserl, due to the operative intentionality the cognising subject does not *construct* objects (by *imposing* categories on experience) but rather *enacts* and *inhabits* them, by separating them from the lived experience.

Notwithstanding the *immediate* feeling of certainty that intuition brings, there is a *tension,* since the intuitive moment with all its fullness of certainty is also realised as a *prehension,* i.e. the "inadequate or imperfect grasp of something, where the content of the grasp adumbrates or points to something beyond what is given. … a form of incomplete cognitive apprehension" (Tragesser, 1977, pp. 18-19). This tension is resolved, at least temporarily,[75] with the application of the intuitive object into the task, asking legitimacy in "*determinately ordered systems of possible appearances*". Mathematics, with its abstraction and formalisation,

> But no matter how indeterminate it may be, it is still a pointing ahead to a bodily shape, to a bodily coloring, etc. And only appearances that adumbrate things of that kind and that determine more closely what is indeterminate in the framework of this prefiguring can be integrated concordantly; only they

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[74] The extract is from Merleau-Ponty's notes on Husserl's *Origin of Geometry*.

[75] Although the issue is not my primary concern, it would be interesting to compare with Sophocles' Oedipus trilogy for the Lacanian interpretation of Oedipus's fate, as a *reoccurring problem*, first with Sphinx, then with the plague that forced him to take his sight away. Oedipus says in the last part of the trilogy that the problem is never solved in a final manner and that it signals an ongoing process. For an analogue in mathematics education research, see the latest work of Liz de Freitas with Nathalie Sinclair (2014, p. 560), where they argue for "bridging the concrete and the abstract".

> can stay the course of an identical *x* of determination as the same, being determined here newly and more closely. This holds time and again for every perceptual phase of the streaming process of perceiving, for every new appearance, only that the intentional horizon has altered and shifted. … To our mind, the aspects are nothing for themselves; they are appearances-of only through the intentional horizons that are inseparable from them. (Husserl, 2001, pp. 42-43)

The "cognitively striving subject potentially making a decision" strives for fulfilment, since this tendency for fulfilment is indigenous, as "belonging to the structure of intentional lived-experiences as such *without the ego coming into play*". "A remembering or an expectation already stirring in the background" are already endowed with affords and gains of the intentionally structured horizons, that bring them to the state of an *intentional lived-experience,* due to operative intentionality.

By tracing the origins of intentionality (by exploring the "intentional horizons") we apply a reduction that leads us down to the constitutive origins. Thus we have a unique chance to take a glimpse at the silent world of perception, the passive stage and its operative intentionality, which is revealed due to the concomitant manifestations of reflection. The reflection is rendered effective, due to the *intentionality of act,* which is the constant theme of mathematics education research, highly neglecting the part of operative intentionality. Husserl's genius distinguished between these two intentionalities (e.g. 1999a, p. 98, also cf. 1969, 1970a, 1973, 1989, 1991, 2001), and put effort in studying the structure, the genesis of both and their links, with his constitutive and his genetic phenomenologies.

A final remark has to do with intuition's first appearance, which is synonymous to original perception, renewed every moment "of the streaming process of perceiving, for every new appearance" (ibid., pp. 42-43). It is in the flow of appearances that the hyletic material appear, as they are intuitively grasped, where *inner* and *outer* horizons interplay in ever new forms,[76] with new intentions ("the intentional horizon

[76] For the *inner* and *outer* horizons see the two following sections.

has altered and shifted" (Husserl, 2001, op. cit.)), which are supported (fulfilled) by new intuitions.

> Proper to every appearing thing of each perceptual phase is a new empty horizon, a new system of determinable indeterminacy, a new system of progressing tendencies with corresponding possibilities of entering into *determinately ordered systems of possible appearances*, of possible ways that the aspects can run their course, together with horizons that are inseparably affiliated with these aspects. In the concordant *coinciding of sense,* they would bring the same object as being ever newly determined to actual, fulfilling givenness. To our mind, the aspects are nothing for themselves; they are *appearances-of* only through the intentional horizons that are inseparable from them. (ibid. p. 43)

The more or less unmediated *originality/immediacy* remains as a genetic seal in intuition's more elaborated forms, of mathematically "*determinately ordered systems of possible appearances*", in the learning activities that are analysed in this research.

1.4.6 The objects' self-giveness and allure – *Intentionality of act*, made possible due to the *operative* intentionality.

What is crucial from our cognitive point of view is that the operative intentionality makes possible the *intentionality of act* (Merleau-Ponty, 2002a, p. 486), "which is that of our judgements and of those occasions when we voluntarily take up a position" (p. xx). I have already mentioned that the *intentionality of act,* which will be the theme of other sections, is the prevalent theme of mathematics education, while systematically missing the embodied aspect of cognition or taking it as ready-made, under an objectivist, instrumentalist approach of experience. What is important from our cognitive point of view at this point is that the intentionality of act is responsible for the "more complete apprehension of the object" (Tragesser, 1977, p. 18), as it is achieved through its application to the task, shaped as a *key idea*, but this is not the whole story. and actually being based on *intuition*. But the original *prehension*, comes with an intentionality other than the intentionality of act,

the first incomplete comprehension (ibid. p.18) comes with the intentionality that precedes the intentionality of act, already bringing with it the certainty that an objectivity is there, a certain candidate for protentive intentions to be fulfilled, before they'll acquire names due to the intentionality that follows, the positing intentionality of act. By the interplay of the two intentionalities an object is born, as it appears to the *lived body* through the operative intentionality, and as it consequently acquires an object status, or an object *label* if you like. The embodied, original intentionality opens up a field of exploration for the *self-given*[77] actualities and potentialities of the emerging object-like entity, which is transformed to an object according to what is at task. While the first intentionality is responsible for our *immediate contact with the real,* the second one is *already an abstraction,* as it meets its *first idealisation* by subscribing to the discursive realm (Husserl, 1936, 1970a; Merleau-Ponty, 2002b).

The two intentionalities are two poles that are interrelated and mutually fed, which is what supports—in the Husserlian theory—the interrelation and the mutual effect of empirical and abstract intuitive objects

- e.g. symmetry intuition and what it triggers in Mary's case (Part 3, §3.4.2.3.),
- the double (in two stages) x^2 categorial intuition and how it is triggered by the figure that empirically sprouts from the 5 points, or
- the final quality of number 10, which started as the distance of the fixed point from the wall and it evolved to a variable, as it was fed by its continuous abstractions,[78] while it was retaining an embodied kinesthetic aspect up to the final step.

which thus surfaces as a candidate for constitution, for becoming an object, posed as an object by the intentionality of act.

I see the same road over and over again in my way to work at 7:35 am, and I realise an intentionality that makes all these cars move in accordance to the traffic

[77] I.e. as it appears, as it is given in the lived experience, also cf. Index for *self-giveness*.

[78] Before it became a variable it was used in the construction of the first formula, and due to the process that was followed there number 10 was reflected on 6, for the construction of the second formula.

lights, and their motility to be more coordinated than usual. A reflection on the presence of these cars and the way that they move at this time of the day, combined with an imaginary variation on every previous working day, reveals a critical *invariable* (an *essence* in Husserl's terms), which is the transportation of most of these people in order to go to their work space; it reveals it as a primary intentionality, which runs through their driving related actions. There is no doubt that most of these people intend, as I also do, to arrive at their work space as early as possible. Each and every driver that is involved in this process at 7:35am has her own personal traits in dealing with driving, with work related stress and all others issues in her life-world that are somehow involved (each time) in her working mornings at 7:35am. Still, this intentionality finds expression in the motility and the regulation of the traffic, as I see it every working day; but it is not a reflective intentionality nor a discursive one, despite its exceptions, or rather its variations, as my case is. Despite behaviors that are based on reflection, this intentionality starts as an intersubjective, pre-reflective prefiguring of the way that each car moves, and each driver's actions are broader manifestations of their traits and the special conditions they meet each time, resulting at the 'more regulated' traffic that I encounter every working day, and almost *only* at that time.

This intentionality leads to acts, thus it becomes intentionality of act, shaped in many different subjective forms. It starts as operative intentionality, since it operates in the lived body, since it starts from driving a vehicle, a largely habitual activity that in many cases becomes an extension of one's living body or, rather, a re-constituted living body, *shaped* as a car sort of speak.

Instead of pursuing the collective intentionality, manifested in the previous example, I focused on individuals, in order to bracket anything but the following triangle, and explore its operations on the ground of the learners' lived experiences:

- ego pole –
- subjective –
- object pole

(mentioned as ego-cogitatio-cogitata in Husserl (1970a, p. 207 – see the same page for Husserl's inversed Cartesianism).

The links between the learner and her constitutive accomplishments (between the *ego pole* and the *object pole*) are shaped by the *intentionalities* of consciousness, namely the *operative* and the *intentionality of act*. They are intertwined and mutually affective, and, although the huge issue of affection, as a diluting and fertilising agent of acts and actions (e.g. Husserl, 2001, Ch. 2) is beyond the scope of this study, it may suffice to say that Husserlian bibliography offers a fair exploration of the topic. See for instance Husserl's *Selbstgebegenheit,* rendered as "self-giveness", meaning "both the giving of the self of the object on the part of the subject as a noetic process,[79] and the self-giving of the self of the object from the object" (Husserl, 2001, p. lii). Husserl's self-giveness concerns the object, which is not only a passive receptor of the ego's attention:

> it exercises an affectively significant allure on the perceiver to be constituted as such, that is, for the 'ego' to tum toward it attentively and to constitute it as a theme of interest in an active manner. The fact that *something* is actually heard or seen or smelled, etc., is due to 'affective rays' radiating from the object, drawing in its wake the horizonal referential implications. (ibid. p. lii).

Husserl is focusing on the "the ipseity of the object, the self-givenness of object it-self in the intentional relation" (Husserl, 2001, p. lii), which is the relation that will explored in the study's three cases.

[79] That is a process directed by the *ego pole* towards the *object pole*, having the specific intentionality of objectivation, i.e. the intentionality to release *objective sense* out of transcendental and immanent objects that appear in the Life-world of the subject.

1.4.7 Retention and retentional modifications of the primordial impression – The continual synthesis of identification as a preparation for the constitution of the object – Inner and outer horizons of the object's appearance

We saw how the protentions (the empty intentions that concern what is to come)[80] play a part in pre-figuring the lived experience. But there is also the history of "the mode of original intuitability" (Husserl, 2001, p. 217), of the primal impressions which become past but they are not lost, although they do "not go undiminished" (Husserl, 2001, p. 217). As Husserl puts it, "[w]hat is given in the mode of original intuitability, of having a self in the flesh, givenness in the flesh, undergoes the modal transformation of the 'more and more past.'" (Husserl, 2001, p. 217).

It is a trait of intentional consciousness to retain the past impressions (Husserl, 2001, p. 114), not as mental representations but as *temporally* determined modifications of their original moments (Husserl, 1991, pp. 243, 244). For the object of perception to appear—firstly as an *object-like* formation—a *continual synthesis of identification* is necessary, for which the retentional consciousness is decisive:

> The constitutive process of this being modified in consciousness is a continual synthesis of identification. What is given to consciousness is continuously the same, but it is pushed back further and further into the past. … the emergence of the new primordial impression is continuously linked up with the impression that has immediately passed and is modified into the form of the just-past. (Husserl, 2001, p. 217).

In other words, "the retentional process is a process of identifying synthesis—the sameness of the objective sense runs clean through it" (ibid. p. 219). It is a first, pre-reflective unification of primordial lived-experiences, which are modified as 'moments' of the identical object. What is seen as identical, already from the beginning has got other sides,[81] hence *inner horizons,* as well as unexplored *outer*

[80] Husserl (2001, p. 112) also calls protentions *memories of the future*.

[81] See section 3.3 for the protentive anticipations, i.e. the "empty indications that realise new perceptions".

horizons,[82] since it appears in a *context* that becomes gradually fulfilled, which remains to be revealed, and is under the state of *determinable indeterminacy*. The aforementioned state is also part and parcel of the retentional process, which "strives for clarity" (Husserl, 2001, p. 472) and depends on *closer determination,* due to the *inner horizons* of the object.

From what we have said already,

> we see that every perception *implicite* invokes an entire perceptual system; every appearance that arises in it implies an entire system of appearance, specifically in the form of intentional inner and outer horizons. We cannot even imagine a mode of appearance in which the appearing object would be given completely. No final presentation in the flesh is ever reached in the mode of appearance as if it would present the complete, exhausted self of the object. Every appearance implies a *plus ultra* in the empty horizon. (Husserl, 2001, p. 48)

Before the reflective, thinking process, there is a 'preparation' by temporal and qualitative configurations of the raw (*hyletic*) empirical material. It is truly essential here that Husserl 'projects' the real as not only the validation, the grounding and the source of the material that will be objectified, but also as the *active partner* to the ego's constitutions. The *object pole* for Husserl is one of the three terms that constitute objectification and the subjective cognitive life in general (Husserl, 1970, p.217); the produced objects are not categorically pre-determined and even within the habitual modes of activity there is always a choice of novel determinations. The real transcends our understandings of it. And yet it proliferates our lives with objects that are or become active and 'permanent' for our lives, like tables or the Pythagorean Theorem. Because the real is involved, through which objects are separated as soon as they are thematised, and the real according to Husserl always appears in *determinable indeterminacy,* and through shifting intentional horizons. The subject is not isolated from the real, since she lives *in* and *for* the world. The

[82] For Husserl's detailed account of inner and horizons, and their significance for the cognitive lived experience see chapter 12 in the Essential Husserl (1999) and in particular ∬2, 3.

contact between the *subject* and the *real* is *embodied*, since consciousness is already embodied and since the lived body (the empirical ego) of the cognising subject is an active medium of the contact, as an organ of perception. In the subjective contact between the ego pole and the object pole the *subject* enriches—as well as being enriched by—the subjective *grasp* that she has over the object, in a process that could be approached phenomenologically as reactivation.[83]

CHAPTER 5. Objectification, intuitions and the living body – Empirical and abstract intuitions, and the genetic features that render intuitions detectable

1.5.1 The *transcendental ego,* the *living body* and the "intentional background of constitutive accomplishments"

There are different terms and expressions that describe the living body, each one of them adumbrating its particular character: it is the *empirical ego* for Husserl, the point of contact between the subject (the learner) and the world, the *body-subject* (*Leib*) in contradistinction to the body-object (*Kőrper*), it is the *incarnate body* for Merleau-Ponty and the *embodied consciousness* for both (Tito, 1990, p. 185). "As was the case previously with the psychic, everything that has newly flowed in is now concretely localized in the world through the living body, which is essentially always constituted along with it" (Husserl, 1970a, p. 210).

The living body is the new movement that the "empirical human ego" "has become involved" (op. cit.), the zero point (Husserl, 1970a, p. 332), "[t]he zero-region of the right-left, the zero-region of the above-below, the zero-region of front-back" (Husserl, 2001, p. 584), the lived-body (*Leib*), guided by a consciousness (Moran & Cohen, 2012, p. 31). Alfred Schutz gives a fair and concise description of it, as he introduces Husserl's transcendental intersubjectivity.

> How, in the frame of the natural attitude, is mutual understanding *(Einverstandnis)* in principle possible? The answer given by Husserl in *Ideen I* (Par. 53), on the occasion of the preparatory analyses of pure

[83] Mary's case is an example of reactivation.

> consciousness, refers to the experience of a linking of consciousness and body *(Leib)* to form a natural, empirical unity by means of which consciousness is located in the space and time of nature, and which, in acts of 'empathy,' makes possible reciprocal understanding between animate subjects belonging to one world. 'The experiences of others manifest themselves to us,' we apprehend them by virtue of the fact that they find bodily expression. (Collected Papers III, 1970, pp. 51-52)

Husserl's paradigmatic reflection on the "transcendental-phenomenological reorientation" (Husserl, 1970a, p. 210), in his last work that was published before his death in 1938,[84] concludes with the *transcendental dimension* and the *new transcendental discovery* that is at hand for the phenomenologist and for one phenomenologically introduced and interested in its methodic exploration…

> As was the case previously with the psychic, everything that has newly flowed in is now concretely localized in the world through the living body, which is essentially always constituted along with it. I-the-man, together with the transcendental dimension now ascribed to me, am somewhere in space at some time in the world's time. Thus every new transcendental discovery, by going back into the natural attitude, enriches my psychic life and (apperceptively as a matter of course) that of every other. (Husserl, 1970a, p. 210)

To summarise, "everything that has newly flowed in" from external perception is "concretely localized in the world through the living body", and the latter is "essentially always constituted along with it" (op. cit.). The living body is the "intentional background of constitutive accomplishments", it is the *essential* link that

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[84] Due to Husserl's illness it was Eugene Fink's substantial help that the third part of the work was published in Belgrade after the previous two parts had been published there already, since, being a Jew he "was denied any public platform in Germany" (Husserl, 1970a, p. xvii). "We can see that the Crisis was to be an immense work, much longer than the present text. But no manuscript has been found for the conclusion of Part III or for the text of Parts IV and V" (ibid.). Yet Husserl left more than 50.000 stenographic pages when he died, a huge amount of his ever evolving reflections, rescued by Leo von Breda to Louven, Belgium and only recently fully accessed to the public. And it is of no less importance that Merleau-Ponty was one of the first to visit Husserl's archives there.

is most often discarded, since it is silent and not immediately visible. When I am teaching in the class, explaining what I have written on the board and I hear students going "aha! So that's it!", I know that their living bodies are fully operational towards possible mathematical meaning that my performance evokes. Husserlian phenomenology reveals the structure of the original cognitive moment, by employing *intentionality* in a radically new sense (the dirtectedness *of*, which is an immanent feature of consciousness, taken by Brentano) and distinguishing between essentially different intentionalities. The living body (Leib), is where *operative* intentionality functions (e.g. Husserl, 2001, Merleau-Ponty 2002) and where the—more readily detectable—*intentionality of act* becomes possible (cf. Reuter, 1999; Morris, 2012), which is the same thing.

What else do I make out of all of this? That the *synthesis of identification* that starts in the pre-reflective stage of the lived experience of the cognising subject (see Graph 3 in Part 1 §1.5.7), is possible since it is intentionally oriented by and towards a pole, which is no other than the *ego pole,* which *transcends* the objects that she constitutes and survives (transcends) her own past appearances, as an 'identical' ego; therefore it originally is a *transcendental* ego. "The transcendental ego is the *intentional* center of all conscious life and, hence, of all objectifying *experiences*, all affects, all *valuations*, and all *volitions* and actions" (Drummond, 2007, p. 205). Obviously Brown (2012), following Althusser's terminology has hit the nail on the head with his approach of *identifications* instead of *identities*.[85] As far as my scope of analysis is concerned, it is from *the lived experience of identifications* that *constituted* identities evolve, into what becomes the *mathematical interface* of the learner/teacher. Since these identities are under formation, and they may change dramatically,[86] the results of my study appear concomitant with Brown's approach, although deriving from a Husserlian rather than a Lacanian stream of thought.[87]

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[85] "Indeed, in Althusser's terminology, there are no identities, just identifications with particular ideological ways of making sense of the world" (Brown, 2012, p. 72).

[86] See Diana's case for an example of such a change in Part 3, Ch. 2.

[87] For a 'balanced' approach of Husserlian phenomenology from Lacanian and Freudian perspectives see Leder A. (2012).

Brown's idea in a nutshell is that this survival for the ego is a failure for the concept, since the perception of the concept by the subject is never quite identical and it introduces edges that redefine the concept (Brown, 2012). Although I would sign all statements mentioned here as critical aspects of the phenomenological approach to cognition, I would have to add that what makes me feel uneasy is a rather functionalist *epistemic* approach of mathematics, as a result of privileging Kuhn's evolutionary ideas. My study adopts the epistemological historical evolutionary theory of Lakatos instead, since mathematics may not be confused by natural science, (e.g. 'counter' theories may coexist, such as Euclidian geometry, which was considered as the only valid form of geometry for more than 2000 years, but after the discovery and the foundation of the non-Euclidian geometries it did not lose its validity and it became a branch of a broader field). As Grabiner (in *New directions of the philosophy of mathematics,* 1986, p. 212) puts it, mathematics "has at once the least destructive and still the most fundamental revolutions"; which is something that makes mathematics a science of a particular source, and certainly not a positivist science, despite its applications to positivist scientific fields.

And what we are concerned here with is the *grounding* of mathematics in the learners' lived experience and the object constituting activity that takes place in it:

> [C]onsciousness is to be understood throughout as a stream that constitutes objects and that is subject to pure essential laws; and we must understand that this still takes place on the founding level of passivity. For that is the soil upon which the free activity of the ego moves and without the knowledge of which the higher accomplishments of this activity must remain for us completely unintelligible. (ibid., p. 386)

Hence, in order to understand "the higher accomplishments" of the "founding level of passivity", that the object constitution actually is, the attention will be drawn to perception and in particular intuition, since they are "the first fundamental shapes of consciousness that are at issue for the structure of consciousness that is specifically logical; they are the first foundations in the logical structure that must be situated and understood" (Husserl 2001, p. 607).

1.5.2 Summary of objectification and of the distinction between Husserlian and constructivist approaches – Intuition as objectifying act

Phenomenology, as the movement inaugurated by Edmund Husserl, has had a major influence on contemporary theory through Husserl's published work and its influence on Heidegger, Gadamer, Arendt, Levinas, Sartre, Merleau-Ponty, Derrida and numerous other philosophers and scholars (Moran, 2000). There has being a surprising convergence between many deconstructive, analytic, and critical theory readings of Husserl (Welton, 2000) that produced a standard picture during the 1960s and the 1970s, which continues to be dominant today. "[S]everal postmodernist interpreters have added a second critical analysis that reduces Husserl to being a foil for developmental and genealogical accounts that attempt to overturn the very possibility of transcendental phenomenology" (Welton, 2003, p. xi). Husserl's name has resurfaced more frequently in recent decades "because the main concept of his unfinished later philosophy, the 'life-world', increasingly draws attention" (Held, in Welton, 2003, p. 4). In my research, phenomenology is seen as reviving our living contact with reality; and the Husserlian approach to intuition that will be exemplified here is seen as a critical way of approaching the learning praxis, distinct from misinterpretations based on Kantian and Cartesian perspectives.

The concept of intuition is stressed as an important feature of mathematical learning activity in mathematics education, and the bibliography on this topic is particularly rich (Beth & Piaget, 1966; Hersh, 1997; Davis & Hersh, 1998; Fischbein 1994, 2001; Poincare 2007). Most approaches of intuition are using a *constructivist* lens, originating in Kant, Descartes or Plato; the general idea is that processes that inhabit the *cogito,* the discursive human mind take over, and the abstract intuitive processes are ultimately settled in internal, relational cognitive models. Intuition borrows its materials from a world that is either internal (in the case of abstract objects) or external (in the case of empirical objects). Descartes' *cogito* insight has been confused with the Husserlian approach to intuition, as Hintikka (2003, pp. 178-179) explains:

> Among other alleged similarities, Descartes' *cogito* insight has been compared with Husserlian intuitions. … For Husserl, the certainty of an intuition is due to the fact that an object has been given to me, not to the fact that its object is in my consciousness. In contrast, according to one especially interesting interpretation of the Cartesian *cogito,* the certainty of my thinking is created by myself, viz. by performing an act of thinking.

On the other hand, Kantian approaches to intuition see it as a mental act that presents empirical or mental objects to us, based on pictorial models, ideality, abstractness, absolute perfection and universality (Fischbein, 1994, 2001). What is common in most approaches to intuitive thinking is that they do not give a clear picture of what intuition actually is and how it emerges. Hersh (1997, p. 65) maintains that "[w]e have intuition because we have mental representations of mathematical objects. … We don't know how these representations are held in the mind/brain. We don't know how any thought or knowledge is held in the mind/brain". Fischbein (1994, p. 88) explains that when a child "affirms that a line may be extended indefinitely, he expresses an intuition. This intuition is related to his experience". Fischbein's *behavioural roots of intuitive representations* (ibid, Ch. 7) are concomitant with the approach of constructivism and its radical ramifications to perception in general (e.g. Richards & von Glasersfeld, 1980; Cobb, 1994; Cobb & Yackel, 1996; Steffe & Kieren, 1994), being faithful to the Kantian perception of experience as knowledge:

> When Kant maintains that the "I" makes experience possible, he is working with a very special concept of experience. By experience Kant means *objective* thinking, *knowledge.* "Experience is an empirical knowledge," writes Kant (CPR 208 [B 218]). Hence for Kant experience is knowledge and already involves objective categories. (Tito, 1990, p. 78)

The radical constructivist turn attempted to part from the ontological theoretical commitments to the Piagetian cognitive pedagogy, by classifying them as *trivial* (Steffe & Kieren, 1994, pp. 720-721), as it intended to release the possibilities for expansion of the Piagetian claim for cognition through action. Although this is a

necessarily rough description of a rich movement that broke from behaviourism, empiricism and naïve idealism (cf. Steffe & Kieren, 1994), it may suffice for the space available to summarise by adding that constructivism never broke its ties with Kantian transcendental idealism, as it is in the core of its perception of individual cognition.

In contradistinction to Kantian and Cartesian approaches the research adopts a Husserlian perspective, where "there is a level of experience that has not yet been subjected to the objective categories, a level of experience that is the ground of the objective categories" (Tito, 1990, p.78). This level is orchestrated through what Husserl and Merleau- Ponty call the *living body,* "which is never absent from the perceptual field" (Husserl, 1970a, p. 106). Husserl also distinguishes between *operative intentionality* and *intentionality of act,* and Merleau-Ponty's phenomenological analysis of perception is crucially using Husserl's operative intentionality (Reuter, 1999, pp. 70-72). The operative intentionality is "already at work before any positing or any judgment" (Merleau-Ponty, 2002, p. 498), and it makes the *intentionality of act* possible. The intuitive process, starts from the separation of forms that are imposed on the raw empirical material. "What is given in intuition for Husserl are components of reality, not our impressions of them, … at this absolutely crucial point he remains totally foreign to the Kantian way of thinking" (Hintikka, 2003, p. 174). In this sense, intuition is approached as an essential mediator between the learner's world-as-lived and her objectification process. The objects, in order to become objects for the cognisant's embodied consciousness (i.e. to become consciously realised as such) they are intentionally synthesised from pre-objective (object-like) formations, in novel thematisations, and thus objectified. These constitutions of objects bring with them profiles of the real, through processes of 'I can', 'I could', rather than 'I think'. Processes where intuition is responsible for bringing direct evidence for empirical or abstract constitutions of suitable objectivities for the tasks *at hand*.

This spiral description concerns the motor drive of the achievement that objectification is, which is no other than the fulfilment of intentionality. Objectification takes place when intuition spots at last what was implicitly sought

already, it re-crognises it (cognises it anew) as fitting, fulfilling the web of intentions that were attracted to it, yet effected by the allure of the objects themselves. For phenomenology does not hand in the creative aspect of knowledge to the subject pole alone or to her constructions. Rather, it attributes significant importance to the object poles that are 'constantly refreshed'. I move in the kitchen as I am 'forced' to notice a small dark discrepancy on the white kitchen counter to my left; I almost have to turn and look. It is my dark red pen. Oh yes, the one I was looking for. Equally well I'm drawing on the arsenal that I am afforded with when I am after a mathematical task, according to particular features that I choose to focus my attention; and it is most usually in these areas that I gain results.

This genetic feature of the fulfilment of intentionality in the objectification process is also important for this study, since it is a precursor of the engagement of intuitions; since the only features missing from objectification as it was just described, in order to be realised intuitively are immediacy and the feeling of certainty ("filling of clarity") that inevitably comes with it, due to the fulfilment. Intuitions in all these states play a crucial part, seen from the Husserlian perspective that is advocated here, since they make possible the association of distant neighbourhoods of the learner's world of objects, and enabling new thematisations of previous objectifications. and by what it makes visible as a possibility to explore further. And this does not necessarily mean that what is discovered or unearthed (de-sedimented) is known in advance, but rather that it is intended and prepared pre-reflectively, before its appearance as an object. The interplay of the ever present indeterminacy of perception—i.e. the absence that co-determines the presence of consciousness [88] in transcendental (empirical) and abstract (mathematical) objects—demands that the evidence of the first prehensions (Tragesser, pp. 18-19) of the object, of the first incomplete apprehensions that intuitive processes bring to the surface with the filling of certainty be lit by action, in order to *be known* for what they are, for their *objective sense* that will be specified

[88] Husserl's term for the co-presence of what is absent is *apperception* (cf. Glossary/Index of terms). See also Anthony Steinbock's (2003), approach of Husserl's *home-worlds* and *alien-worlds.*

in a certain judgment.

> When we speak of belief, of a consciousness of being, we indeed have in mind within the lived-experience in question, a directedness, an intending directedness toward the object. *Doxa* is intending. Intending can mean: *I* intend, *I* am directed toward something in a believing manner. But already within passivity we find an intention, as belonging to the structure of intentional lived-experiences as such without the ego coming into play as a subject who is [actively] directed, as a cognitively striving subject potentially making a decision. It is what in truth transforms a concrete lived-experience, perhaps like a remembering or an expectation already stirring in the background, into an intentional lived-experience. If, from the ego, this same lived-experience later becomes patent, the intention takes on the act-mode of the *"ego cogito."* Intending, understood in its current sense, signifies something wholly determinate, something we obtain in a completely clear manner when we think of our analyses of experience; it is that intending that is ultimately an intending constituting a unitary objective sense, and that makes up the fundamental structure of a consciousness as a consciousness directed toward an object. (Husserl, 2001, p. 449)

Husserl gives here indications of the subtle but crucial distinction between his approach to experience and Kantian/Cartesian ones. It is where immediacy (the direct, instant manner) will meet intentionality (directedness towards an object) in the intuitive act.

For Husserl, objects are not perceived in isolation but against a background and in the midst of a surrounding world of other objects and also of other *living bodies* which are also other persons, animals, and so on (Husserl, 1989). The object's life starts in the pre-objective, pre-reflective sphere of conscious (yet not reflected) raw impressions (Graph 3 in Part 1 §1.5.7), rendered intelligible through the living body, and integrated by the *embodied* consciousness, as the object is separated from the cognisant's lived-experience, and *learned as it is used* for the intention that brought it to life in the first place. This is the story of objectification

in a nut-shell, from its intentional origins and the subjective way that it is taken up—as a conscious striving for fulfilment of the *intentional* embodied consciousness—up to the achievement of the object constitution. Now, when this fulfilment takes place with *immediacy* an *intuition* is realised, as the 'aha' feeling, as a response to a gut feeling, releasing a *feeling of certainty* for the outcome, for the intuitive object, as the suiting fulfilment of the intentionality that brought it to surface.

Of course for Husserl, what I just described is only the first stage of his five stages objectification process (1970a, pp, 359- 361). But we need not concern with his full theory of objectification here, since the aim is to look at the *emergence* of the object within the learner's investigation, as something standing against (Gegenstand) the flow of appearances in his consciousness (Radford, 2003, pp. 39-40; cf. Graph 3 in Part 1 §1.5.7), due to answering (fulfilling) a particular intention that the learner starts with or develops. The fulfilment of this intention with immediacy is the intuitive object, which can further the exploration of the learner and become a key moment for her investigation, as we will see in the analysis of the data.

1.5.3 Two kinds of intuition and the categorical structure of knowledge

Intuition is a highly controversial topic in science and philosophy and some authors even suggest that the concept of intuition may be misleading and that it should be avoided in scientific reasoning (Fischbein, 1999, p. 11). Many researches describe different kinds of intuitions that are supposed to function on different levels of cognition, from empirical to abstract knowledge. Piaget distinguishes *empirical* intuition—which concerns the physical or psychological properties of objects (Beth & Piaget, 1966, p. 224)—and *operational* intuition that "possesses an immanent logic" (ibid., 212). Fischbein (1994) classifies intuitions by their origins as *primary* and *secondary,* where "Primary intuitions refer to those cognitive beliefs which develop in individuals independently of any systematic instruction as an effect of their personal experience. (p. 64), while "the category of secondary intuitions implies the assumption that new intuitions, with no natural roots, may be developed. Such intuitions are not produced by the natural, normal experience of

an individual" (ibid., p. 68).

Let us look more closely to the philosophical base of Piaget and Fischbein's positions, since the latter's approach is also endowed with a *constructivist* lens, originating from the Kantian theory of mind (Otte, 1998). For Fishbein intuition is an act of consciousness that presents empirical or mental objects to us, based on pictorial models, ideality, abstractness, absolute perfection and universality (Fischbein, 1994, 2001). In this approach intuition appears in an already structured, thematised world, as a "combination of incompleteness of information and intrinsic certitude" (Fishbein p. 51). The 'blindness' of constructivism to the pre-objective stage of the cognitive lived experience makes the constitution of abstract entities incomprehensible: it leaves the structure of the lived world unexplicated and the origins of constitution are lost in predisposed categories.

> For Husserl there is a level of experience that has not yet been subjected to the objective categories, a level of experience that is the ground of the objective categories. The 'I' is such an experience, an experience that is preobjective, and is not knowledge. (Tito, 1990, pp. 78-79)

Husserl (1970b, 1983b) gave a privileged position to intuition for systematic knowledge (Smith, 2007), by placing it at the core of his *principle of all principles* (Husserl, 1983a, p. 44). He approached intuition as that act in which a person grasps something immediately in its bodily presence and also as a primordially given act upon which all of the rest is to be founded. For Husserl the source of intuitions is reality itself, constituted by the subject not in a metaphysical manner but through the immediate contact with it, the intuitive contact. "What is given in intuition for Husserl are components of reality, not our impressions of them, ... at this absolutely crucial point he remains totally foreign to the Kantian way of thinking" (Hintikka, 2003, p. 174).

1.5.4 Husserl's Stages of intuitions - Empirical and Abstract intuitions

According to Husserl, the intuitive process starts from the separation of the forms that are imposed on the raw empirical material (the *hyletic* data or simply *hyle,* deriving from the Greek word for matter), from the data themselves.

"Empirical intuitions are due to reality impinging on my consciousness, not to the activities of my own mind. ... In order to reach empirical intuitions, I have so to speak to look outwards, not inwards" (Hintikka, 2003, p. 178). Husserl calls this first stage *Anschauung*—intuition—and the outcome of this stage is an empirical object. For Husserl the empirical intuitions are not an *experience* in the Kantian sense, since consciousness is already embodied. The separation is not performed due to the categories that manipulate the empirical (sensory) data; nor is it a mere mental recognition of patterns that are so far familiar only to others. It is intuition that allows *embodied intentional consciousness* to distinguish objectlike entities from raw material and transform them into empirical objects. The objectilike entities reside consciousness until *operative intentionality* will call them forth to a new thematisation, a new unity that is constitutive for the new object. Embodied consciousness is what Husserl and Merleau-Ponty call the *living body,* "which is never absent from the perceptual field" (Husserl, 1970a, p. 106, also cf. Merleau-Ponty, 2002a, p. 87, and Tito, 1990, p. 185). The body[89] plays an essential role in the fulfillment of *operative* intentionality, since it is "a sensory-motor behavior through which the world is constituted for man (sic) as the world of human consciousness" (Merleau-Ponty, 1964, p. xvi). The *operative intentionality* is "already at work before any positing or any judgment" (Merleau-Ponty, 2002a, p. 498) and it "produces the natural and antepredicative unity of the world and of our life" (Merleau-Ponty, 2002a, p. xx). It is pre-reflective but this does not mean that it is unconscious (Reuter, 1999, p. 76); and it can be revealed:

> We do not see the operation, since it operates. We see what would be missing without it; we circumscribe it as what makes speech be a "speaking of" ... and not be the conscious having of the idealities implied in speech. (Merleau- Ponty, 2002b, p. 44)[90]

Due to the operative intentionality the cognising subject does not *construct* objects (by *imposing* categories on experience) but rather *enacts* and *inhabits* them, by

[89] The body as (intentional) subject, not as an object (cf. Reuter, 1999, pp. 71-72).
[90] The extract is from Merleau-Ponty's notes on Husserl's *Origin of Geometry*.

separating them from the unstructured raw material. Despite the immediate feeling of certainty that intuition brings there is a tension, since the intuitive moment with all its fullness of certainty is realised as a *prehension,* i.e. the "inadequate or imperfect grasp of something, where the content of the grasp adumbrates or points to something beyond what is given … a form of incomplete cognitive apprehension" (Tragesser, 1977, pp. 18-19). What is crucial from our cognitive point of view is that the operative intentionality makes possible the *intentionality of act* (Merleau-Ponty, 2002a, p. 486), "which is that of our judgements and of those occasions when we voluntarily take up a position" (p. xx). The intentionality of act is responsible for the "more complete apprehension of the object" (ibid. p. 19) that is achieved through its application to the task.

The other crucial novelty of Husserl's approach towards Kantian epistemologies that concerns us here is his introduction of the *categorial intuitions*, which are intuitions of *abstract* objects (Husserl, 1970b). The categorial intuitions take place in two stages, (structurally) following empirical ones and (factually) using their products.

When the second intuitive stage takes place—the *categorial* stage—an abstract object emerges, in a process that is *"more like seeing than imagining'* (Hintikka, 2003, p. 177). According to Husserl:

> Husserl's essences, do not come to us separated from the empirical objects whose "forms" they are. ... This intuition has the effect of opening to my consciousness, not only an empirical object consisting of matter and form, but that form or essence in itself, separated from its *hyle.* (Hintikka, 2003, pp. 177, 180)

The *intuitions of essences (Wesensschau)* are a special case of *categorial* intuitions and I refer to both of them as *abstract intuitions*. For the same reasons *Wesensschau* will refer to both of them.[91] Let us sum up the two main kinds of intuitions according to Husserl:

[91] Examples of categorial intuitions and intuitions of essences are given in the data analysis section.

- The *empirical intuition (Anschauung)* enables the subject to extract forms-objects from the raw empirical material (the hyle) through her lived experience, due to operative intentionality. The disentanglement of forms turns them into empirical objects and they become known as such, as soon as they are separated.
- The *abstract intuition (categorial,* and *intuition of essence,* or *Wesensschau)* enables the subject to extract abstract forms-objects from the products of empirical intuition, as well as from the results of more elaborated objectifying acts. The disentanglement of abstract forms takes place due to the intentionality of act, and turns them into abstract objects that are still based on the subject's lived experience.

1.5.5 Intuitions as interrelated forms, functioning in a similar way

The stages of the categorial intuition, i.e. the empirical and the abstract stages are linked, although the second one may immediately succeed or take place long after the first one.

Since the empirical objects of the first category can become materials for the second one that produces abstract objects, Husserl provides a strong indication for the emergence of abstract objects as *interrelated with empirical objects,* which either are or they are related to empirical material, previously objectified. And he approaches cognition as linked to the learner's *embodied consciousness*, rather than to *pre-existing objective categories.* He grounds the seeing of 'essences' on an embodied perception of conscious activity, and identifies intuition as the operator "on the overlap of my consciousness and reality" (Hintikka, 1995, p. 82), as an actual mediator between the world-as-lived by the individual and her constitution of abstract mathematical objects. His inquiry concerns "the way in which the life-world constantly functions as subsoil, into how its manifold prelogical validities act as grounds for the logical ones, for theoretical truths" (Husserl, 1970a, p. 124).

Categorial intuitions follow the pattern of empirical intuitions: The latter intend objectlike entities and by intuitive synthesis thematise them into empirical

objects; the former collect invariable features (essences) of empirical objects and they synthesise them into abstract objects.

Husserl's theory elucidates how an empirical object is transmuted to an abstract object during the 2nd stage-category of intuition, how an abstract object undergoes the same process as it is transformed into another abstract object and so forth. Husserl, as a mathematician, is aware of the particular nature of *mathematical* norms and *mathematical* objects which can proceed higher and higher in a process of abstraction.

1.5.6 Properties/features intuitive moments

Intuitions have three principal properties/features, emanating from Husserl's theory and allowing us to detect and classify them as such. These properties have a *genetic* (existential) link to intuitions, since each one of them is a 'component' of the possibility for the intuition to emerge as such:

- *Intentionality,* namely the directedness of consciousness towards objects, which is for Husserl a fundamental attribute of all conscious acts, especially in their 'pregnant' state (Husserl, 1980, p. 106), as intuition certainly is. Intentionality is the property that makes intuition an objectifying act: "what makes seeing an essence an intuition is not that it is seeing an *essence,* but that it is seeing an *object* which is 'itself given'" (Hintikka, 2003, p. 181, italics in the original).
- *Immediacy* is the intuitions' critical feature, which distinguishes them from other concepts such as memory or imagination (Husserl, 1970b; Hintikka, 2003; Held, in Welton, 2003). Immediacy is understood in a double sense of instantaneity and of the actual straightforward involvement in the world, i.e. a sense of
- The *feeling of certainty* ("filling of clarity", Husserl, 1970b) is a feature that comes inevitably as the object emerges, that is as the object fulfills the intentionality that brings it to the surface. It is nothing but the filling of intentionality, expressed as the feeling that 'this is the sought object', felt with immediacy.

These criteria were used in the analysis of the empirical data allowing me to understand the different kinds of intuition that the students used, according to their relations to the students' lived experience. The Husserlian theory of intuitions offered a solid explication of their origins in the cultural and time dependent world of experience, and it shed light on their expansion to more elaborated intuitions.

Being attentive to visual and kinesthetic evidences of the *sensory-motor* pre-objective experience of the student I expect to have a glimpse on how intuition transforms raw material into objects, thus contributing something critical for mathematical education by better understanding the constitution of the mathematical objects by the students.

1.5.7 A summarising table and an explicative graph

The Table 1 and the Graph 3 that follow give an overview of the Husserlian theory of perception and cognition that my study employs in the analyses of the learning episodes in Part 3. It is read from top to bottom and from left to right, and the 'final product' appears at the bottom right box. Also, the last column is running parallel to the structure of the other three columns, being its constant source. The Table 1 makes use of Husserlian texts from (Husserl, 1969, 1973, 1999a, 2001). The quotation marks also refer to Husserlian texts (1999a, 2001). Although *passivity* and *activity* appear in the diagram as separated they are depicted as such only for the sake of the analysis. As Husserl (1973, p. 203) put it,

> When we distinguish two levels of interest and, corresponding to these, two levels of objectifying operations, viz., that belonging to receptive experience, on the one hand, and that of predicative spontaneity, on the other, this distinction of levels should not be construed as if the different operations were somehow separate from each other. On the contrary, things which must be treated separately for the sake of analysis and which, genetically, are recognized as belonging to different levels of objectification are as a rule actually closely entwined.

The *operative intentionality* and the *intentionality of act* that are linked to the passive and the active stages respectively are "closely entwined"; they belong "to

receptive experience, on the one hand, and that of predicative spontaneity, on the other", thus corresponding to the "two levels of objectifying operations" (ibid.).

The fact that the operative functions precede the active, constitutive activities does not mean that the former are in fact something independent from the latter; or, as Husserl put it, "[t]hat receptivity precedes predicative spontaneity does not mean that the former is in fact something independent" (ibid.). Moreover, and most pertinent to my study, these intentionalities operate with corresponding *intuitions,* as they proceed towards *objectification*. A new intentionality transforms the sense that I had endowed my table, as soon as I notice its beechen drawers; by thematising them I recall that I have a beechen armchair at the basement, one that I had long forgotten. The levels of receptive experience and of predicative spontaneity succeed each other, yielding objects in their entwinement. And "only that can be originally predicated which has been originally given in an intuition, apprehended, and explicated" (ibid.).

The Graph 3 contextualises the Table 1, as it places the learner's cognitive acts within the broader life-world of her mathematical activity. It depicts the general structure that links *objects* and *tasks*, *objects* and *object-like formations*, and *object-like formations* and the *primordial lived-experience,* i.e. the three temporal stages of the present 'moment'. Finally, it relates the functions of the *active* (pertaining to actions) and the *operative* (pertaining to operations) syntheses, as leading to *objects* and *object-like formations* respectively. Both the Table 1 and the Graph 3 could be functional as points of reference during the analyses of the three learning episodes, as well as for the general discussion of the outcomes of this research study.

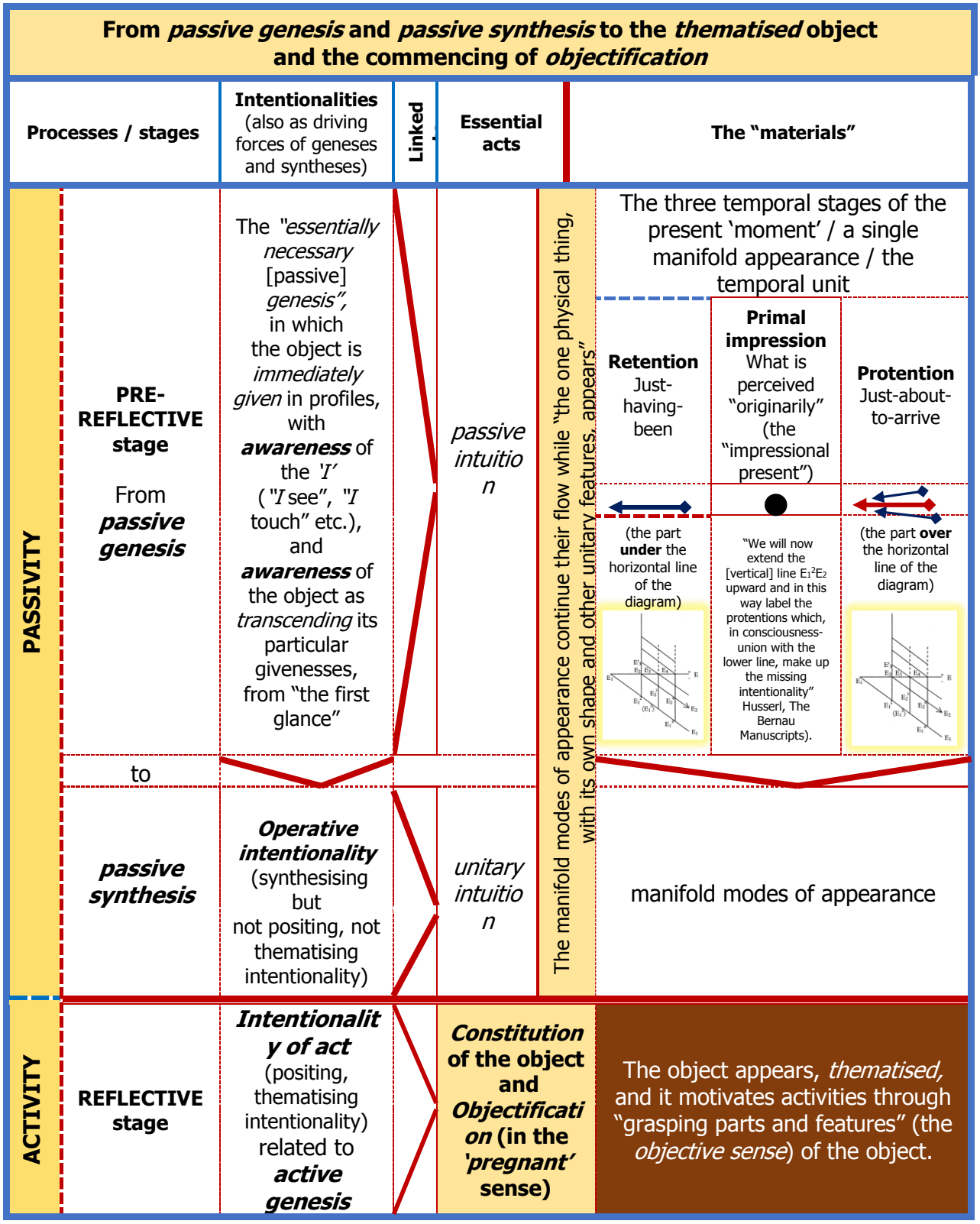
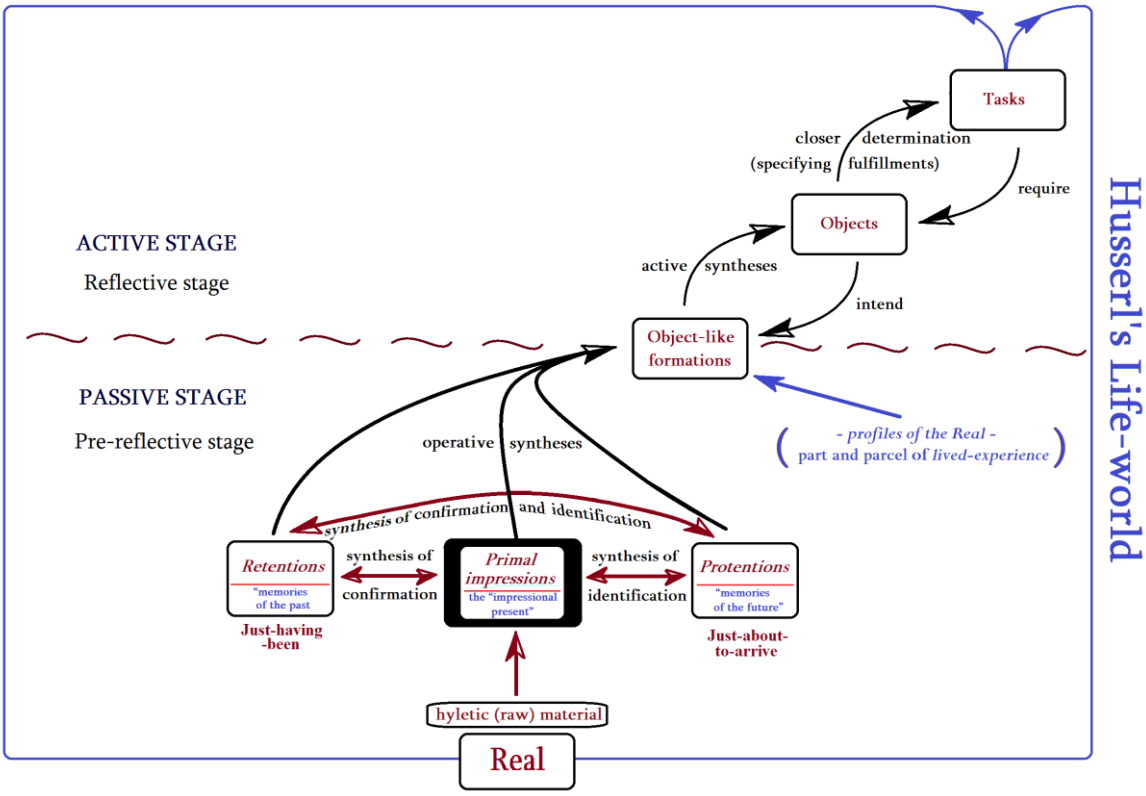


Table 1: From *passive genesis* and *passive synthesis* to the *thematised* object and the commencing of *objectification*.



Graph 3. Husserl's cognitive acts within the life-world. The early stages of objectification.

PART 2 – Methodology/Methods

A few words for the following sections of this chapter

As we need to know how the theory that we were implicated with in the previous part of the study will help us understand the analysis of the data that follows, this chapter aims at delineating the methods used and the overarching methodology, their close connection to the chosen theoretical framework and the inevitable choice of the methodology, as the only one that could afford the findings of the study. More than a tautology, the expression just used indicates the critical part that the methodology played for the accomplishments of this research study.

A summarising diagram is used (§ 2.2) and the general methodological routes that are necessary before delving into the data are given, for each of the three cases. Since each case is studied by focusing on the *diversity* of expressions that are met particularly for that case, rather than on the commonalities across the cases, which are expected to arise only after the data will become findings. Only after immersing in each students' mathematical accomplishments (shaped as mathematical objects), and after arriving at the phenomenological description of each case, with its emphasis on the intentional character of conscious acts, may we be able to argue that the methodology/methods employed here were justified, post factum. As anticipated by the phenomenological approach of the study, the latter is faithful to the students' thoughts, acts and strategies, as I implemented a novel approach to mature Husserl, one that resorts to concrete data in order to bring evidence of the constitutive accomplishments of the students, and the stimulus of these intuitive ideas for theoretical—mathematical—actions. While the transition from empirical to abstract contexts is also under investigation. Bearing in mind that all these actions are seen as phenomena, under the phenomenologist's special attention (specific "regard") that its principles are delineated here.

Hence this part of the study concludes with the methodological implications of a historical-philosophical issue that lies at the core of the philosophical research questions of the study, also related to the question concerning the possibility of

knowledge, known as Meno's paradox. The introduction of this issue is also expected to deepen methodologically the theoretical distinctions that were clarified in the previous part of the study, between my phenomenological approach and other theoretical approaches to learning—the cognising praxis.

2.1 Introduction

The driving force of this study is my intention to explore the genetic moments of cognitive activity, in the sense that Husserl (1936, 1989, 1973, 1991, 2001) gives to it and Merleau-Ponty (1964, 2002a, 2002b) develops it to the primacy of perception. I aimed at describing the mathematical phenomena as they occurred in the tasks, from the students' individual perspectives, and to start from a viewpoint free from hypotheses about what the students' constructions should be and as free as possible from preconceptions (Husserl, 1970b). For this reason I attempted to apply a phenomenological reduction that focuses on the experience in which the objects in general and the mathematical objects in particular are given to the cognising subjects, instead of analysing the objects of experience themselves. Under this perspective, bracketing was necessary in order to approach the students' objects *as they appeared*. Starting from 'bracketing' the body of the mathematical tradition and 'bracketing' cultural interventions in the learning process I was driven towards "*bracketing* of questions regarding the actual existence of the experiences on which we reflect and the object of those experiences and focusing instead on the object just in the manner of its *appearance*" (Drummond, 2007, p.160). I participated in the whole course of learning sessions and I adopted the teacher's strategy (who withheld directing the students), in order to draw on embodied experience to proceed through empirical intuitions to abstract intuitions. Inevitably, I was not a detached and impartial observer but an internal participant who tried to enter into the perceptions of the students and see the learning situations as they saw them. It was an ongoing and adjusting process of bracketing and coming to the phenomenological attitude (Moustakas, 1994; Lester, 1999). In this context the current study approaches the learning process from a point of presence within and through it, attempting to make it as transparent as possible by accessing its origins

in the student's conscious acts, deeply rooted as they are in the individual and common horizons of the world-as-lived and perceived by the learner.

2.1.1 From the communicable expressions to the *intrapsychically constituted structures* – The methodology and the methods used in the study

Under the perspective that I adopted during my participation in the sessions (as it was delineated in the previous section), and while I was aiming at the investigation of the students' *key ideas,* I explored the student's communicable expressions (verbal, figurative, diagrammatic, etc.); different levels of systematisation, generalisation and abstraction surfaced, as I tried to gain access to the "intrapsychically constituted structures" (Husserl, 1936, p.163), from which these expressions originated. Through the students' communicable expressions of their *key ideas* I came to *intuitions* as critical acts for the emergence of mathematical objects. Hence my focus was specified, from objectification 'in general' to the study of the role of *intuition* in the process of objectification.

By applying the phenomenological reduction I was "led back" from natural beliefs to the reflective consideration of intentions and their objects" (Audi, 1999, p. 405). My argument is that the treatment of the particular case studies could gain generality towards the mechanisms of objectification that they exemplify, not by inductive conclusion but by phenomenological *evidence* that "covers other acts besides simply perceptive seeing", while "[s]ense-formations whose nature it is to exist as subjectively produced results are 'grasped' originaliter[92] in being produced" (Dorion Cairns cited in Husserl, 1936, p. 13).

It is in this sense that I achieved evidences of different kinds of intuitions, which were revealed not as empirically existent and observable, but as the residue of my phenomenological analysis. The foregoing approach resulted in a methodology that was consistent with the phenomenological attitude, i.e. with my "leading back" from

[92] The term 'originaliter' is used by Dorion Cairns and Husserl in order to express the 'grasping' in its 'personal' actuality; another term used by Husserl for the same reason is *originarily* (cf. Moran & Cohen, 2012, p. 264 and the Glossary/Index).

natural beliefs to the reflective consideration of intentions and their objects" (Audi, 1999, p. 405). Thus the phenomenological instruments used in this research were as follows: the phenomenological attitude, which was the overarching phenomenological instrument that developed through the study, the phenomenological reduction, and the bracketing or epochè.

2.2 Delineating the methodological instruments that are used by this research – The methodology as the movement from the natural to the phenomenological attitude

In order to apply the theory that was introduced in the theory chapter, it was necessary to use a phenomenological methodology, which was realised through the *phenomenological attitude.* The latter approaches the world as a phenomenon in the phenomenological sense, in contrast with the *natural attitude,* which involves a presupposed commitment of *belief* in the existence and reality of the objects of the experiences in question (Husserl, 1983a, § 30). This distinction was crucial for the application of the theory, since it is one that distinguishes *phenomenology* from *phenomenalism,* as Husserl (2001) explains.

In order to enter the phenomenological attitude it was necessary to apply the *phenomenological reduction*. The phenomenological reduction, which is *"the name for the process by which we move to the phenomenological attitude"* (Audi, 1999, p. 405) involves a "suspension of judgment" (ibid., p. 373), a *bracketing* that Husserl (1999a, § 8, p. 20) calls *epochè*. In other words, entering the phenomenological attitude entails, apart from and alongside the reduction, the putting out of action or *suspending* all the intentions and convictions of the *natural attitude*. But it is important to note that "this does not mean that we doubt or negate them, only that we take a distance from them and contemplate their structure" (Audi, 1999, p. 405).

Another term that is used by Husserl for the aforementioned suspension of the *general thesis*[93] of the natural attitude (Husserl, 1983a, § 30) is *bracketing,* and it is the term that I will mostly use for the *epochè*. The image of bracketing presumably comes from mathematics, where the expression within the brackets can be kept separate from the operations going on outside the brackets. Bracketing is not a negation, but rather like putting something in quarantine, a putting out of use, a 'switching off' of the activity of the thing (Moran & Cohen, 2012, p. 52). But bracketing does not mean to deny my prior knowledge of the analysed experience and my prior knowledge in general, since both sources of knowledge are needed in order to understand what is happening in the first place and in the last instance; bracketing is rather to put into question all that appears, in exploring the how of their appearance; to use my prior knowledge of the experience in re-cognising the objectification process, as it becomes meaningful for the person who *constitutes* it as sensible, who discerns objectivity in her constitution, from the viewpoint of the person, and using my suitable prior knowledge that is related to what the person's constitutions within the given task are, according to what is at task for her each time. Deactivated in quarantine, my knowledge is activated again at the point of relevance, or be discarded fully, as interfering with my reduction to the radical questioning that I'm undertaking.

Moving from the natural attitude, with its naïve, "unquestioned belief in the world" (Moran & Cohen, 2012, p. 49), to the phenomenological attitude—which examines the world as a phenomenon—by using the method of the phenomenological reduction,[94] corresponds to my movement from the data sources to the data (Graph 4); and in the following sections I will clarify the elements of these relations.

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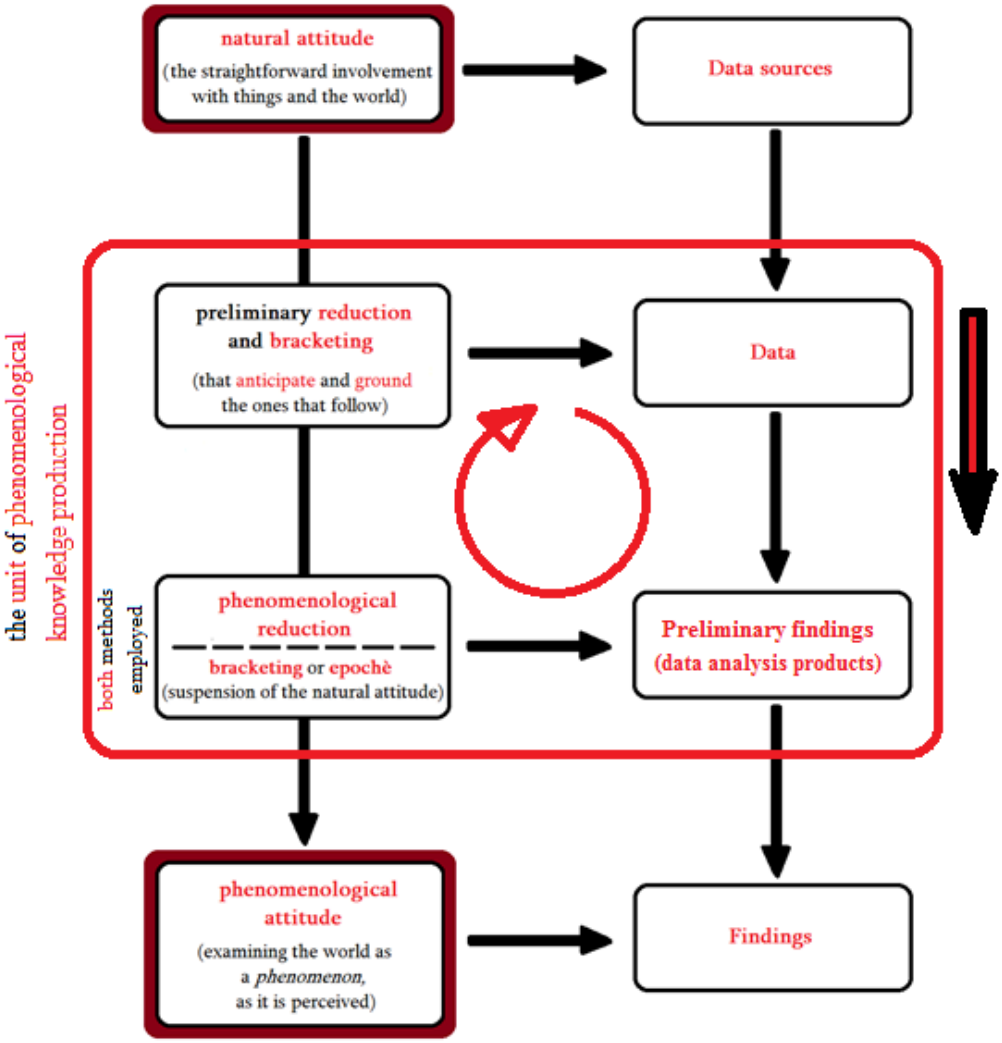
[93] "Thesis" in Greek means position, and Husserl speaks of it as 'positing' *(Thesis, Position, Setzung)* (Moran & Cohen, 2012, p. 136). For Husserl's description of the suspension or bracketing of the naive acceptance of the general thesis of the natural attitude cf. Husserl, 1983a, § 31, pp. 59-60; 1999a, §8.

[94] And including *bracketing* or *epochè* (Moran & Cohen, 2012, p. 110).



Graph 4. Summation diagram of the phenomenological methodology

2.3 The three settings and the corresponding data sources

My aim when I started this research was to study objectification, since I wanted to explore and understand better how the students conceptualise mathematical phenomena, and how they produce knowledge in order to tackle the problems that they encounter during their learning activities. The bracketing that was mentioned in the previous section was a specifically phenomenological one, since it bracketed everything but the shifting of my attention from the object to the manner in which the object is apprehended by consciousness (Moran & Cohen, 2012, pp. 52-53). It is in the same sense that the data analysis is a reduction to the critical acts that

determined the judgments of the students, when they were 'forced' to devise their own mathematical expressions. As my investigation progressed, intuitions became the key in unlocking the objectification process; intuitions presented by Husserl in their *objectifying* and *embodied* nature. And my methodology consisted in *putting out of play* all aspects that did not allow me to focus on objectification as a *phenomenon*, as a challenged existent of subjective acts *in* and *for* the world, not as a ready-made social-cultural transference of information.

As my research progressed and the phenomenological reduction and bracketing were employed in more and more areas of the learning episodes, new data were appearing as associated in one way or another with the analysed ones, or as products of previous reflection and analysis themselves, and at the same moment as 'raw material' for a novel thematisation. The process of the flow of data that were transformed to findings due to the phenomenological methodology, and the latter being applied to the findings, i.e. to the products of the analysis, is depicted in the diagram with the frame that focuses on the pair of real and intentional components of the study. In this iterative process where the flow of data was transformed to findings, according to the theory, intuition became the key in unravelling the students' conceptions and understandings. The analysis, the methodology and the theory converged in triggering findings from the data. But in order to arrive at the appreciation of intuitions as the critical feature of the abstract mathematical objects' constitutions by the students, it was essential to bracket all aspects that intervened and blurred my view, such as the sociocultural cognitive aspect. My bracketing did not negate or cancel the sociocultural influences; it rather parenthesised them (Husserl, 1983a, §31, pp. 59-60) in order to focus on the phenomenon in question, from the first-person perspective.

My data sources concern 3 settings from the course that took place at the academic year 2010-2011, set up by a certain teacher, whom I will anonymise. These settings are the *classroom,* the *students' coursework* and the *lengthy interviews'* settings. The settings were distinctive clusters of data sources from which data could be selected. The data sources for each of these settings are as follows:

- Classroom **setting**:
    - Audio recordings of the sessions.
    - Some video recordings of the sessions.
    - My observations during the sessions.
    - The students' written reflections during the sessions.
    - Short interviews during the sessions.
- The students' coursework **setting**:
    - The documents that the students submitted as their coursework, at the end of the sessions.
    - Written reflections during the sessions.
- Lengthy Interviews' **setting**:
    - Recorded interviews that took place after the end of the sessions.
    - E-mails that followed up the interviews.

My data sources were not phenomenological when I collected them, since they were 'open' to any approach and any interpretation. In order to transform the data sources to data I made certain choices, in a process that involved reduction and bracketing.

Firstly, I chose the first person perspective. I made this choice because I wanted to be faithful to my phenomenological perspective, as I perceive it, since I wanted to explore the students' understandings from their point of view. Wishing to gain a better understanding of the students' objectification process as it appeared to the learners, I would not be able to do it if I would adopt any other perspective, like for instance the third person perspective. In the latter case, I would have examined the phenomenon in question as a natural scientist does; I would have created a distance between me and the students' experiences, which would have led me to quantifying and measuring them. The perspective that I chose, i.e. the first person perspective put the learner (the first person) at the centre of investigation. My aim was to 'look over the shoulder' of the learner, as she objectified her key ideas, in order to tackle the problems she encountered during her investigation.

Secondly, I bracketed my convictions concerning mathematics. I had been teaching mathematics for more than 15 years in Greece, and it was my first year in England, having close access to a very different educational environment, where the emphasis is put on problem solving (Brown & Coles, 2012) and high achievement in international competitions such as TIMSS and PISA (Brown, 2012), in contrast to the more traditional and theoretical emphasis that is the case for the Greek educational system. For instance, I did not use my mathematical knowledge in trying to predict what the students were trying to find, although I inevitably used it as I tried to recognise what they were doing. In other words, my mathematical knowledge was important in the *understanding* of what the students were doing, but I did not *judge* their methods as effective or not, according to my previous experience as a mathematics teacher, but I rather attempted to re-cognise what sort of achievements their objectifications signaled, from *their* viewpoint. My views about the nature of mathematics and as a teacher of mathematics were important, but they were not in focus.

Thirdly, the teacher helped me bracket the 'traditional' teaching part in the students' objectification activities, since his teaching was based on the limited direction of the students and the promotion of their confidence in posing and addressing their own questions. It is not implied of course that the teacher put teaching per se out of play, since he was actually exercising his teaching style; but he certainly removed issues that would be taken for granted in most everyday institutional form of teaching (schools, universities, training courses etc.), namely the guidance of students towards particular mathematical 'answers' and specific paths in order to acquire them, or the deployment of his knowledge of and about mathematics and the usefulness of the teacher's methods. The courses did without all this,[95] in order that the students themselves would take the initiative. If the

[95] These features as well as others, such the openness in legitimising the students'' methods and results, were strictly kept by the teacher throughout the courses, and my process was an adoptive one, as I was realising that everything I was doing was somehow interpreted by the students and possibly influencing their strategies, as they wanted to know more about it—the teacher had introduced me as a Greek teacher of mathematics who is doing research on the students' understandings of mathematics.

students would act as *protogeometers* (see Part 1, §§ 1.2.2, 1.3.1) or just follow ideas of others in their groups was totally up to them. But what was most important from my research perspective is that the teacher's methods were opening space for the learners in acting out the tasks as *they* chose, and the according space for my study to acquire 'material' for reflection.

Another critical feature of the whole project was the reflective character that the teacher brought to the sessions, in the form of short (2-5min) reflective breaks. This practice was added smoothly, as a 'useful practice for all', and the students were quickly adjusted to it. While the reflective practice developed with the sessions up to the end, when the students were expected to add a reflective aspect to the assignments (coursework) that they delivered to the teacher. What also became a valuable data source for me was the students' written reflections in their coursework (useful in all three cases that will be analysed), and those classroom reflections that I managed to gain access (see the first case in this study). The reflections facilitated the students to evolve their strategies and even their attitudes towards the sessions (see the second case in this study), and they helped me to

- Track important 'details' of the student's feelings towards their treatments of the tasks.
- Formulate questions and topics of discussion for the interviews that followed.
- Gain a fuller picture of the learning episodes through the more detailed view of the students' objectification 'moments' that these reflections offered.

The students' 'systematised' reflections, combined with the teacher's methodical abstention from guidance and his minimised intervention to the students' treatments of the tasks, were integrated by my study's methodological choices. And they facilitated the emergence of processes that my theoretical filter was aiming at. Thus the methodology was combined with the theory, since the methodology allowed to be discerned what the theory was ready to build upon.

2.4 Common features of the data sources that I selected from the whole of the classroom data sources

Thirteen students participated in the Nature of Mathematics course (hereafter NoM) and 18 sessions took place, which were all audio recorded, and two of them were video recorded. Out of the data sources that belonged to the classroom setting I selected three sessions, and three students, in three different activities. I will now explain what led me to this selection (the preliminary reduction and bracketing), starting from the common features that drew my attention to each of these data sources.

These three data sources had some things in common, which made them important for me:

- They were all considered very important for the students, since they were included in the coursework that they delivered at the end of the course. The fact that the students included their work for these three activities in the coursework also meant that they put their best efforts in them, which is one feature that drew my attention. In particular, Diana's investigation of the Doubling modulo (the 2nd case study) was her favourite of all her investigations, and the same was the case for Mary (the 3rd case study). Ivan (the 1st case study) had more than one activities as his favourites, but the one analysed here was certainly one of them.
- They were all rich data sources, for the following reasons:
    - I got more familiar with these three students than with the most of the other students. One reason for this, in the case of Diana and Mary was that I happened to sit with their groups more often than with other students, although I was trying to share my presence to all the students' groups. Also, Mary and Diana were extrovert and open to me. Their openness allowed me to approach them, to have a deeper look at what they were trying to achieve and get a better understanding to what was happening from *their* point of view. It was not as straightforward with Ivan, since his inquisitive approach

towards me concerning the tasks did not allow me to sit with his group too often, as I was keeping an observatory rather than participatory part during the students' mathematical investigations. Yet, there was a growing mutual respect, between Ivan and I, which paid off in his interviews, and the excellent cooperation we have had even three years after the sessions, concerning clarifications of his work and providing help in order to contact other students of the same class (Mary and Carry), since they had finished their studies in the meantime.

- Each data source was very rich in the sense that in each of the three learning episodes I managed to get very close to an incident where each student struck an impasse, persisted in his or her individual way and finally overcame the difficulty, through mathematical objects that the students constituted for this purpose. I witnessed in the course of the data collection how their state of indeterminacy, confusion or even despair gave results, in each of the cases that were analysed:
    - Ivan, with his insistence for appealing aesthetic forms persisted in his particular method throughout the session, he did not manage to get results, but he had a feeling in the end that he could make it; he went home and he discovered how previously neglected findings could yield a complete solution to the task.
    - Diana got into a state of despair as she could not understand what the other students were doing, and the teacher would not provide any guidance. But she managed to follow her own method—for the first time in these sessions—and she constituted mathematical objects that were different than anyone else.
    - Mary was fully confused during the classroom investigation, having misunderstood what the activity was about. But she persevered when she went home and she unlocked the task in

the most surprising way, going further than the requirements of the particular task.

After the end of the sessions these three data sources stood out for me due to their rich environments, and their significance for the students themselves.

2.5 How the data sources yielded data

Each of the three data sources provided me with data—which later gave me case studies—due to the following additional reasons:

- The data source in Ivan's classroom setting, as it was enriched by the structure of the classroom investigation, his coursework and a lengthy interview, yielded data as follows:
    - Due to a previous session where many diverse student approaches were observed but not recorded, one of the aims of the session that became one of my three main data sources was to record different methods that the students would follow. Since it would be senseless to record different groups simultaneously I decided to record the activity of one of the groups—which happened to be Ivan's, Carry's and Donald's group—and the teacher asked the students to volunteer giving short interviews at the beginning of the second part of the session, after the break. The students warmly agreed and I took seven short interviews, including Ivan's and Carry's interviews. Moreover, as a usual practice of these sessions, there were three five minutes *'reflective breaks'* during the session, when the students were asked to write their current understandings of the task, and summarise where they thought they presently were, in terms of the maths. Most importantly, the students agreed to deliver their notes at the end of the session and take them back after I had photocopied them; thus, all the work that Ivan and his group did during the session, including their reflective texts, became part of the particular data source. As I looked at the written material of Ivan and his group, and I listened to the interviews of Ivan and Carry, I already thought that it was a very

interesting data source, which could give interesting data. In particular, my attention was drawn to the distinctively different strategies that Ivan and Carry followed throughout the session (I was not able to detect any particular method in the case of Donald), and Ivan's aesthetic approach to the task. The conflict between Ivan's and Carry's methods was not resolved—despite the exchange of ideas between them—and this was particularly interesting for me because from my experience as a teacher I am aware that this is a case that most often takes place in a mathematics classroom, and the difficulties in communicating different approaches prevail, either between students or between the teacher and the students. But there was still a problem concerning my interest in objectification and the fullness of the particular data source, since none of the students had managed to overcome his or her impasses during the session. Two months later, Ivan's coursework came to my hands. It was the moment that the particular session data sources—i.e. my observations, the recording of the group's activity, Ivan's and Carry's short interviews, and the photocopied material from the group's activity—became data, since Ivan, alone from his group, had shown a fervent activity at home, depicted by his detailed reflection, and culminating in an interesting objectifying moment that yielded amazing results. This data source (i.e. Ivan's coursework) gave me further insights into Ivan's thinking, crucially complementing my preliminary observations of the aforementioned data sources. A lengthy interview with Ivan followed, which focused on the aforementioned investigation at university and at home, and unfolded the richness of his particular objectification. Finally, my e-mail communication with Ivan, which extended up to three years after the end of the course, clarified important issues that had remained obscure in the initial data sources. It was through the aforementioned process that I collected rich data for the particular investigation.

- The data source in Diana's classroom setting, as it was enriched by her coursework, and a lengthy interview, yielded data as follows:
    - One of the main reasons that Diana's data sources were chosen was that they were a striking contrast to Ivan's, since unlike Ivan, she was struggling throughout the sessions. Although Ivan had difficulties, he was not struggling to the extent that Diana was. During the learning episode—the classroom setting—that I selected it was obvious that she was desperate, although at that time I was engaged in another group's investigation. I could hear her almost shouting that she could not understand how to proceed and how the other students were dealing with the task. I went closer in order to understand what was going on and I noticed that she had written some notes that looked like an algorithm to me. I happened to recognise what Diana had written—as a set of commands that seemed to have acquired the form of algorithms—since I was familiar with computer programming. I thought that something interesting was taking place there and I did not want to intervene; so I deliberately ignored her call for help and I moved back to the group that I was observing at the time. But I kept this incident in my mind as a potentially interesting source of data, wanting to explore further what was happening. Diana, quite unexpectedly, managed to work out a solution, a way out from her despair and confusion. It was another case that interested me as a teacher, since I was aware from my experience that no matter how hard teachers try, nothing can guarantee that their efforts will pay off. I had been a witness of the help that many of Diana's classmates tried to offer, by explaining what they were doing in order to attain results. But she could not understand their ways, and the teacher would not offer any guidance, being faithful to his "limited direction" method of teaching. Diana had to move on her own or give up. When the session finished my only data source from Diana's investigation was my

observation of the difficulties that she encountered, and I arranged a lengthy interview with her, partly because I wanted to understand better what had happened and how she dealt with it. Although the interview took place five months later Diana had kept her notes during the session, and I was surprised to know that the particular task was her favourite one; not only had Diana not given up, despite the difficulties that she had encountered, but she had managed to devise an answer to the task that was different than anybody else's way of tackling it. The objects that Diana constituted became the focus of my interest, and she helped me to take a close look at them by e-mailing me the material from her classroom investigation. The coursework that Diana submitted at the end of the sessions (which obviously included her reflection on the particular task) became another rich data source, and it also became a source of validation of what we had discussed in the interview; and a few more e-mails that she readily sent me helped clarify some issues that had remained unclear in the aforementioned data sources. Another interesting aspect of Diana's solution, which goes beyond the purpose of my research was that her solution would probably be disregarded by many teachers, as not being relevant to what the students were supposed to find; but the course's open-endedness towards the students' preferred strategies and results legitimised Diana's solution, thus enabling her to adopt a new perception of learning and teaching. The aforementioned data sources—my observations during the session, Diana's interview, Diana's coursework, the material from her classroom investigation, and the e-mails that we exchanged—enabled me to collect rich data for Diana's investigation. Once I had seen all this I knew clearly that Diana was another person on whom I wanted to focus as a case study.

- Finally, another aspect emerged in Diana's interview, which augmented the grounds for choosing Diana:

- The last three sessions of the course were dedicated to the students' presentations: each student prepared and presented a task to the other students, the teacher and me. Each presentation lasted twenty minutes and it was followed by a ten minute discussion. Diana's impression from her presentation, as it came out in her interview, was a confirmation of the impact that her previous investigation had on her (due to the particular circumstances of her object constitution, and the legitimisation of her results). This aspect accentuated even further the contrasting character of Diana's data to Ivan's.

- The data source in Mary's classroom setting, as it was enriched by her lengthy interview and her coursework, yielded rich data as follows:
  - In one of the investigations that took place during one of the sessions I participated in a group with Mary and another student. Neither of the two students managed to resolve the task, and Mary in particular struggled with it, and she seemed to be confused even on how to start the investigation. The particular group activity was audio and video recorded (after the students' permission), but my attention was not drawn to it until almost four months later, when the teacher told me that Mary had produced a fine *diagram* for this activity. I then arranged a lengthy interview with Mary, in which I focused on the particular learning episode, trying to gain a close understanding of her transition from confusion to clarity, and the objectifications that might have enabled it. As soon as we had the interview I had the clear *feeling of certainty* that fascinating data could be produced from Mary's data sources. The student's coursework—which included her diagram and came into my hands a few days after the interview—enhanced this feeling of certainty. The indications that were related to important features of my research, and which supported my certainty were the following:

- The learning episode was one of Mary's favourite ones.
- The student's transition from perplexity and confusion to articulateness and lucidity called for an explication of the part that objectification may have contributed to it.
- Mary's treatment involved the constitution of many mathematical objects, which could be thoroughly explored due to the detailed and ample data sources (i.e. the audio and video recordings, the interview and the student's detailed description and reflection in her coursework).
- The student vividly recollected her investigation, despite the length of time that had passed until her interview, expressing clear and detailed articulations and showing openness towards me.
- Mary's treatment of the task included 'balanced' empirical and abstract approaches.
- The strong embodied aspect that was involved in the student's treatment enabled an exploration of the origins of objectification and mathematical abstraction.

In other words, the interview was the first data source related to Mary that gave rich data, and it shed light on the classroom data (on what had happened in the classroom) and on the coursework data (on what had happened when she continued the investigation at home). Finally, I arranged a second lengthy interview with Mary, in order to clarify issues that had remained unclear from the previous data, and she kindly responded to my e-mails, thus allowing me to gain the deepest possible understanding of her investigation, and transform her data sources to rich data.

An example of the actual way that I determined what counts as data for this research study is indicative of the methods used and the results that I achieved: Mary perceived her diagram she drew at home with her embodied sense, as a bird's eye-view of her classroom investigation (Part 3, §3.4.2.2). It was a significant

moment for Mary, since it enabled her a gaze of the maze that the task was for her—who knew not what it looked like what she was looking for—from above, in a totalising yet 'static' view; thus coming to contrast with her 'dynamic' embodied participation to the task in the classroom, where she was a few hours ago. Yet these two views joined together as the student's perception brought them to a conjoined embodied sense, on the graph, on the piece of paper. I understood that the silent lived body had operated as an organ of perception, in the transformation of lines and points of the diagram to class locations. And I wanted to know how and where this embodied, intuitive drive would lead the student. We will see the learner's consequent acts in the analysis, as synthesising on intentional ground, as retaining the sense that they endowed their intuitive objects, and as evolving it to expanding, generalising mathematical expressions. Hence, we will manage to take a closer look to the transition from empirical to abstract knowledge, though (interrelated) empirical and abstract objects respectively. Therefore, anything that stood out for the learner, utilising it in various expressions towards the mathematical ones was data for my study, since it signaled the beginning of the exploration towards the intentional relevance of the new object to the task, from the student's intentional perspective.

2.6 The phenomenological methodology involved in the analysis of the data and the 3 cases

In the previous section I gave a description of how I focused on the data sources related to three students and the corresponding learning episodes, starting from the data sources related to the three settings (the classroom, the coursework and the lengthy interviews). The aforementioned process already involved *reduction* and *bracketing,* since my *attention* towards the data sources that triggered the generation of data was not anymore a straightforward involvement with the events that had taken place and the persons involved in these events.[96] By its nature, attention "first of all presupposes a transformation of the mental field, a new way

[96] As was the case when the classroom data sources were generated; the issue of my own presence in the classroom data sources is addressed in the text that follows.

for consciousness to be present to its objects" (Merleau-Ponty, 2002a, p. 33), in this case my own consciousness, as it attempted to thematise my research object-like formations (the data sources) into research objects (the data). As I reflected on the collected—not yet *phenomenologically* instigated—data sources, and having already the intention to *bracket* anything that would get in my way, between the students' acts and their *intended* objects, my attention was drawn to the students' acts as data, instantiated as such by the act of reduction that my attention entailed.

One of the important criteria of the selection of the data sources that would become data was the control over (in the case of Mary) and the minimisation of (in the cases of Ivan and Diana) my own participation in the learning episodes in question:

- I had already made sure that Ivan's classroom investigation would not in the least be effected by my presence to his group, and the same happened in the case of Diana, as I have already explained in the previous section.
- My presence and participation in Mary's classroom investigation could be monitored by subsequent phenomenological methodological and analytical instruments, as the analysis of the data will manifest. The feeling of certainty that was mentioned in the previous section—in the description of Mary's initial data generation after her first lengthy interview—was also related to a preliminary certainty that my participation in Mary's classroom investigation would not become an impediment in the analysis of her data, by crucially interfering with her acts and their *intended* objects.

2.7 How the data were analysed and became findings – The application of the *phenomenological reduction* and the *epochè,* and the arrival at the *phenomenological attitude*

Since my aim was to explore objectification I started looking at the data for instances where the students constituted mathematical objects in order to overcome their impasses. My reduction was now directed to the students' *acts* that intended these objects, investigating the intentional relation that enabled the objects to surface. In other words I was exploring the mathematical objects as they emerged, focusing on the *intentional origins* of these objects.

I have indicated in the previous sections how my preliminary reduction and bracketing yielded data, where the teacher's and my own participation in the students' object constitutions were bracketed, but now, as I was exploring the data intending to apply the *phenomenological reduction,* and while calling "all knowledge into question" (Husserl, 1999b, p. 61), a more fundamental problem emerged: how one can even start the investigation of knowledge production if "every piece of knowledge that might be chosen as a point of departure is thereby also called into question?" (ibid.). In other words, what is the sense of the phenomenological reduction if its task is unattainable in advance? A radical reflection was needed, in order to address this fundamental *epistemological* issue, namely a reflection that turns around the aforementioned question, by asking if knowledge is actually denied, or regarded as doubtful *in every sense,* when it is "called into question". And the answer to the latter question, concerning my methodological inquiry was addressed by Husserl (1999b, pp. 61, 62) as follows:

> if epistemology is to address the possibility of knowledge, then it must possess forms of knowledge concerning the possibility of knowledge that are themselves indubitable, that count as knowledge in the strictest sense—where there is absolutely no doubt about their own possibility or the fact that they have made contact with their object.

Therefore, in order to apply the phenomenological reduction I needed to address and attain three interrelated *methodical* (i.e. pertaining to method) issues:

- I needed to "consider indubitable cases of knowledge or possible knowledge—ones where knowledge actually reaches, or would reach, its object" (p. 62).
- I needed to drive my inquiry back to the "things themselves", not as Kantian *things in-themselves,* but as they were *intended* and as they appeared to the learners, and related as they were to the learners' lived experience.
- The point of departure needed to be an "attitude focused upon life-world phenomena" (Husserl, 1970a, p. 174) and upon the constitution of "things" or thing forms. At the outset I needed to refrain from simply assuming that

> some example of knowledge is indeed knowledge; in this attitude *only experience itself is indubitable,* as it occurs and is simply being reflected upon. This focusing on the learning episodes as life-world phenomena, with the application of bracketing—i.e. the *epochè* or transcendental attitude—was expected to transform the world that I analysed "into the mere transcendental 'phenomenon'", serving as "transcendental guideline for correlative attitudes on higher levels" (ibid.).

As I indicated earlier in this section, my aim was to explore the emergence of *mathematical* objects and investigate their intentional origins. But as I explicated just now, my methodological inquiry oriented my analysis towards the inclusion of the 'non-mathematical' *origins* of these objects (examples will be given in the next chapter), namely in the *object-like formations* that supported the emergence of the *mathematical objects,* and the study of their links. My methodological inquiry back to the original 'self-evidence' of these objects (Husserl, 1936 pp. 163), as ultimately founded on the overlap of the learners' consciousness and their lived reality (Hintikka, 1995, p. 82) finally afforded me with the starting-points of the constitutive activity (Hintikka, 2003, p. 178) and the end-points of the reduction (ibid. p. 180). In other words, my methodological and theoretical lenses converged at what became the key of my study of objectification, namely the Husserlian intuitions. Finally, my key finding sprouted from the integration of my methodological demand and my theoretical "principle of all principles" (Husserl, 1983b, p. 87/151), which lit different modes of the intentional fulfillment of genuine concrete experience. It is in this sense that my phenomenological attitude enabled my findings by actually rendering them visible, from the passive to the active areas of cognitive engagement, due to my converging methodological and theoretical lenses.

All the foregoing exposition concerns the left-hand side of Graph 4 (Part 2 § 2.2), which informed the way that the data analysis was conducted. The bottom-up process that followed, and the constitution of the three profiles (i.e. the data concerning the three students) as three cases for my research, is the topic of the next section.

2.8 The methodology used in the context of the data analysis that follows – The Husserlian method of overcoming the Cartesian doubt, in service of the main argument of the study

In order to clarify a principal claim of my method I need to explicitly distinguish the Husserlian phenomenological reduction from the Cartesian doubt. The radical question that was set in the previous section, concerning the investigation of knowledge when "every piece of knowledge that might be chosen as a point of departure is thereby also called into question", is in principle a Cartesian question, that starts its query from the fundamental doubt that there is nothing in one's world that could be automatically accepted as truth, not even one's own senses. The result of the Cartesian doubt posited *thought*—the *cogito*—as the indubitable ground of all knowledge and cogitatio/cogitationes, the object(s) of thought as existing beyond doubt. Husserl treats the Cartesian question in a new manner, starting with his perception of the Cartesian doubt as a latent, unfinished query for the intentional act (noesis) and the intended object, as intended, and its peculiar profile (noema) that stands out as it is separated for use. Husserl thematised three "headings", namely the "*ego-cogitatio-cogitata*: the ego-pole (and what is peculiar to its identity), the subjective, as appearance tied together synthetically, and the object-poles" (Husserl, 1970a, p. 171). Instead of resting on the indubitability of the thinking ego, he decides to pursue the three headings "one at a time and in *an order opposite to that suggested by the Cartesian approach*" (ibid., emphasis added). He starts with "the straightforwardly given life-world, taken initially as it is given perceptually: as 'normal,' simply there, unbroken, existing in pure ontic certainty (undoubted)" (ibid.). Husserl's new beginning is "the being of the *cogitatio,* of experience as it occurs and is simply being reflected upon" (Husserl, 1999b, p. 62).

> Here the *Cartesian meditation on doubt* provides us with a beginning: the being of the *cogitatio,*[97] of experience as it occurs and is simply being

[97] Object of thought, more precisely of "experience as it occurs and is simply being reflected upon".

> reflected upon, cannot be doubted; the intuitive, direct apprehension and possession of the cogitatio is itself a knowing, the *cogitationes*[98] are the first absolute items of givenness. (italics in the original)

This answer allowed my analysis to move in a productive way, with a certainty that supported the findings of each stage of my recurrent analysis. My acquisition of the primal genetic 'material of abstraction' had found its end point, concerning this study. "[T]he cogitatio is itself a knowing", and abstraction is already the next move, since what comes after the cogitationes of immediate lived experience is the idealisation of the following judgments; "every judgment has a closing [Abschluss] in itself; it is in itself something thematically independent. And yet it is a member of an open and, according to ideal possibility, constantly widening thematic complex, which therefore is not closed" (Husserl, 1973, p. 214). Irrevocably non-constructivist, Husserl deals with the openness and the potentiability (potential and ability) of lived experience, refusing to yield reason to the logicians; he traces the constitution of *logical predicates* in *lived experience* and in the powers of the *lived body,* while unfolding the constitution of further abstractions when he writes:

> Independent unities constantly arise, but only as judgments of a higher level, founded on those of a lower level. Hence, *every theoretical unity of judgment must be identified as a single moment of a higher order,* which, in its much greater complexity, is founded in judgments which are again founded in judgments, and so on. (ibid., italics in the original)

Inspired by such a frame of lived experience and being interested in the theoretical truths that in our case are the mathematical ones, I reflected more and more on the data, as I collected and analysed them. Each of the three cases that will be presented in the next Part of the study was formulated as its 'fragments' were 'coming under the light' of the theory. Husserlian terms were entering one by one the dressing process of the students' strivings. From the researcher's point of view there is a mutual interplay of clarification and exemplification between the

[98] Multiple tense of cogitatio (see the previous footnote).

theoretical ideas and their application in the data analysis. The foregoing is a general description of the transformation process of data sources to data, and of data to more analytically elaborated data (findings), in a cyclical process (cf. Graph 4, Part 2 § 2.2), through successive phenomenological reflections.

The teacher's practices remain bracketed throughout the process, since I focused on the students' acts, and since the latter were not affected by these practices, due to the teacher's own care. A similar method was followed in my own case of participatory observation, since my observatory aspect was stressed and the participatory one followed the teacher's non-guidance strategy, although not practiced as masterfully as he did. What was important for me was the open and friendly access to their groups and their activities; and one of my passports was that I was registered as 'someone who understands mathematics'. This generous access was opened by the teacher's warm introduction and was given to me by all the students throughout the sessions, without any problems.

Overall, *intuitions as objectifying acts* is a suggestion for another approach to intuitions and objectification in our field, and the exemplification of this suggestion is precisely what is at task in the ensuing analysis. All the cognitive 'moments' that will unfold in the following Part of the study are instances when abstraction (shaped as objectification) is shown as it is being constituted for and by the learners. The constitutive process starts from empirical material, which is further integrated by the lived body as embodied consciousness, and tethered into the task, and finally posed as an object worthy for exploration, due to reflective conscious actions. It is a bottom-up process that is unfolded, after the intentional origins of the mathematical objects are traced and explored sufficiently (phenomenologically).

The three learning episodes were shaped as three different adumbrations of the same phenomenon—intuitions as objectifying acts—each case with its own particular features, its own character. The generalisation of my observations will come when, after delving into the lived empirical account of the presented mathematisations, we will see each case been transcended by common features with other cases, which have different character. In other words, when in reflective

variation we will sense the "indifference with regard to actuality which is generated in variation, whereby what presents itself as actual acquires the character of an arbitrary example, an indifferent point of departure of a series of variations" (Husserl, 1973, pp. 348-349). Which is a quality that makes generalisations possible, from particular singularities. In the micro level, such case is the intuition of essences, as well as the more general case of the so called categorial intuitions, each presented twice in the following sections. Such is also the case, namely of singularities that reveal generalities, with the general sense that my phenomenological research is expected to endow its particular cases,[99] due to the theoretical and methodological choices that enabled the particular in-depth analysis.

Inspired by the Husserlian frame of lived experience and being interested in the abstract truths that are the mathematical truths in our case, I reflected more and more while the data were collected and analysed; that is analysed over and over again, shaping each case with the theoretical concepts that fitted well to data sources, which thus became data. Each of the three cases is an investigation standing in-itself, while they all adumbrate the same teaching rules of non-intervention or guidance, which remain unnoticed (bracketed) as we focus on the students' acts, and as the latter are not affected by them.

And most importantly, these three cases support the same argument concerning the importance of intuitions in the objectification process, of all sorts of intuitions, bearing objects and object-like formations that are vital for the learners' mathematical explorations. Intuitions being the instigators of the learners' key ideas, being vital even when they are erroneous, for our understanding of the diversity of the cognitive praxis, as an expression of the living body and its powers.

Intuitions as objectifying acts is a suggestion for another approach to intuitions and objectification in our field, and the exemplification of this suggestion

[99] This issue occurs from the philosophical set of the research questions and is dealt with in the general discussion and the conclusions; in particular, in my response to issues of *validity* and *generalisation* of this research study, which rather than being pre-conceived they emerged as evidence of mathematics-in-the-making and as the exemplification of theoretical terms in the phenomenological sense.

is precisely what is at task in the ensuing analysis. It is all these moments that will unfold in the following Part of the study, when abstraction (shaped as objectification) is shown as it is being constituted in a process starting from empirical material, integrated by the lived body as embodied consciousness, and tethered into the task as it is posed as an *object*, worthy for exploration, due to reflective conscious actions. It is a bottom-up process that is unfolded, after the intentional origins of the object are traced and explored sufficiently (phenomenologically).

The three learning episodes were shaped as three different adumbrations of the same phenomenon—intuitions as objectifying acts—each with its own particular features. And the methodology achieved precisely this: to clear the view of the fundamental process of objectification by successive reduction and bracketing, and lay bare the key role of intuitions in the process, in particular during the preparation of the object, in the pre-objective, pre-reflective stage and its transition to its (constituted) objective state.

PART 3 – Data analysis

CHAPTER 1. Introduction to the data analysis

3.1 Introduction – critical questions that the research addresses

Mathematics is the deeper layer of this study, as well as the form, the *telos* of the learning episodes that will be analysed. The form of the learning episodes is itself an important topic, since I decided to limit myself to the first year of my data collection, especially after I employed the Husserlian theoretical and methodological approach that is deployed here. There are many things to be said, and those selected to be written were considered as necessary and adequate in order to shape an argument theoretically, analytically, methodologically and empirically. An argument that derived from the data as I was delving in them in order to bring to the surface the *kernel* that allowed me to see them as findings, and delineate the origins of this kernel theoretically,[100] and analytically in the ensuing data analysis. My approach to the discussions of the data analysis that follows is framed by a question that was already set in Part 1, §1.3.1, which I want to elaborate more fully here, as we are ready to look closely at how the students found what they originally did not know.

3.1.1 The origins of the subject of the thesis in Meno's paradox

The actual subject of this research is addressed widely in the long history of human cognition, going back as far as Aristotle and Plato, and the pre-Socratic philosophers like Parmenides, Heraclitus and the Pythagorean school, who sought the origins of understanding and of the phenomenal being[101] in different forms of Logos, rather than in mythological entities and genealogies. It is clearly stated in Meno, Plato's

[100] Cf. the process of thematising that I have exemplified in the theory chapter (Part 1, Ch. 3).

[101] The Greek term for the phenomenal being is φαίνεσθαι (phenesthe), which has the same root of the word phenomenon. The φαίνεσθαι is juxtaposed to είναι (being), in different ways for each of the aforementioned philosopher or philosophical group.

dialogue, in a haunting question for education in general and mathematics education in particular:

> Whoever tries to limit the spiritual light to what is at present before the mind always runs up against the Socratic problem. *"How will you set about looking for that thing, the nature of which is totally unknown to you? Which, among the things you do not know, is the one which you propose to look for? And if by chance you should stumble upon it, how will you know that it is indeed that thing, since you are in ignorance of it?"* (Meno, 80D.). (Merleau-Ponty, 2002a, p. 431, emphasis added)

This is the age-old existential question concerning knowledge, and the starting point in answering it—according to Husserl—is that perception always finds an answer, since "perceptual consciousness is not an empty box" (Husserl, 2001, p. 606) and the answer that it provides is part and parcel of the always ("necessarily") incomplete presentation of any object that we encounter with our *lived body*; we always sense a first object envisaging it as a whole, even when we do not have a 'name' for it, and our anticipations concerning the object that we encounter go beyond what we actually perceive; we co-presentify, without thematising it the building where the room that we currently are in is located. But we will thematise it as soon as an intention to leave the building will appear. Everyday perception of public and private affairs is a flowing reality for the cognising subject, performed by relentlessly constituted objects, each of them in its intentional horizons and particular ontic meaning, narrowly or widely thematised. This theoretical clarification is linked with my methodological and methodical choices to the extent that what one determines to look at determines also certain features of the methodology and methods that are adopted; therefore, the reduction, the bracketing and the phenomenological-transcendental attitude, seemed to be the natural methodological milieu in which the research could be accomplished; since it sees lived experience through a multiplicity of concepts that aim at bringing the genetic features to the surface. Only at the moment of the analysis of the episodes will be apparent why the theory insisted on certain concepts like the living/lived body, the operative intentionality, the determinable indeterminacy as a stimulus for action,

the protention as an igniting feature for new knowledge to appear by reactivation of sedimented ready-made objects. Finally, it will be apparent why the better grasping of the detailed Husserlian method, after pure reductions can facilitate understanding the depth of these operative and these constitutive moments. For this reason I will insist a bit more in key theoretical notions that are mutually clarified with the methodological approach, thus adding the analytical background for the section that follows.

Merleau-Ponty delineates the starting point of lived experience, in which one finds what one seeks: one finds what one anticipates to find, but at the same time one realises that there is much more to find. In a true (actual) sense consciousness *is always ahead,* protending in partial anticipation what the apparent (in appearance) sides of the object indicate. This is a source of *new knowledge* for consciousness, as it protends, in not fully-fledged anticipation, what the object could be.

And according to Husserl consciousness is (also) *always behind,* keeping temporal re-presentations alive, as empty indications now that they have not intuitive—immediate—contact with their once lived reality. The latter are called retentions, memories of the past, and they are ready to be presented again if intentionally called for. In particular called for objectifications and the constituted objectivity that emerges with them.

Moreover, to any *constituted objectivity* corresponds an *embodied intersubjectivity,* which belongs to the pre-objective realm of 'I can', 'I sense', rather than 'I think'. I not only perceive this glass on my table by I also can co-presentify how the glass looks from the loudspeaker, to my right; or how my brother sees it, behind and to the left of me. It is an intersubjectivity that is primarily embodied, through the apperception of the other(s) as other living bodies. Thus in the study of the three students' constitutive activities that will be unfolded in the following chapters, they are living bodies expressing their powers on the abstract field of mathematics, in the classroom or at home. Their peers are also present as living

bodies, insofar as they play a role in the analysed object constitutions (even the absence of communication is crucial in Diana's case –Ch. 3).

The teacher's contribution is not denied nor is it prevented from becoming the particular theme of another research study, quite the contrary; the study's residue is expected to become a source of understanding the learners' mathematical constitutions, in which the teacher may extract useful perceptions of her learners' practices, and benefit from a better understanding of their cognitive processes. But it was necessary to bracket the teacher in order to study how the students themselves thought and acted about the tasks; not arbitrarily though, since the teacher himself was practicing teaching in an experienced and deliberate way, with the absence of any form of guidance towards the students.

In the phenomenological methodological perspective adopted by my study the researcher may unfold the *key-moments* in the students' thoughts, as genuinely ("originarily") intuitive, thus revealing the roots of these thoughts in lived-experience. What is at stake in the cognitive experience from the perspective of my research is the tension between the *logical objectivity* and the *carnal intersubjectivity,* which "are related as the founded and the founding" respectively (Merleau-Ponty, 1964b, p. xix):

> the pre-objective order of our carnal intersubjectivity, which is both the founding source of logical objectivity and the measure of its value, begins to exist fully only in the objective order that it founds, which is itself the culmination of the advent of the pre-objective meanings of perceptual experience to explicit existence. (ibid.p. xix)

As it is 'expected', the "objective order" is the only 'visible certainty', the only *sign* concerning the pre-objective realm of the pre-constitutive activity, which Husserl called *passive stage*. Husserl[102] moves beyond any signs in his quest for the origins that lie at the core of the justification of all founded objectivities in lived-experience. And perhaps the transformation of a classroom to an organised collective that

[102] And Merleau-Ponty follows Husserl to a great extent, up to the end of his work.

produces new objectivities by promoting intuitive expressions is an answer to promoting *mathematics in the making,* as my research advocates for.

The experience itself for Husserl includes the (inner and outer) horizons of the acts that we perform and of the objects that we constitute for the tasks at hand, that is within and through the experience of the flowing present, which is irreducibly embedded in the world, more precisely in the life-world of the learner herself.

A thought finds an object "being ahead of itself", a thought has an adequate a-structure (non-structure) for the hosting of a new object. The experience-in-itself is non-objectifiable and each attempt corresponds to an idea in the Kantian sense; but what is really important is the relation of origin that the intuition holds to the real, allowing the indeterminacy of the perception of the real to regulate the symbolic forms, the signs and the artefacts. For, artefactual thought (Kiera & Santi, 2013) is primarily a thought *constituted as such,* by an embodied subjective intentional move, which grounds the (constituted) nature of our own 'divisions' and instrumentalisations.

The *symbolic* realm may need actual experience without any resource in actual experience of the *embodied* character of the *constituted* objectivities, yet, there are certain embodied aspects involved, inevitably: "the x^2 is a *concrete* metaphor for me", replied the known geometer Aggelopulos V. (2012) during a presentation of the historical course of geometry, when he was asked what the empirical material of his intuitions are. Thus, what we may call abstract may acquire a deep transcendental sense for someone who knows geometry in all its expressions, not less transcendental than an object like a chair (Tragesser, 1977), and that is only an example of different levels of abstraction that are based on abstract material that has become empirical (hyletic).

What is unique in the history of epistemology is that there is a *science of experience* based on *phenomenology,* as Husserl envisaged it in his endless reworking of his research, with his ever new introductions to phenomenology, up to the Crisis (Husserl, 1970a) and his enormous unpublished work. This science of experience allows abstract intuitions, and moreover they are based on empirical

ones which, in their turn, are based on pre-objective ones. As a result of his theory more and more *abstract* intuitions are developed. And more hyletic material is produced after the introduction of the first intuitive product, which is an object of a new category—therefore justifying the term categorial in another, significant sense. What is unique in the history of epistemology is that a theory dares to become engaged in the lived-experience, and it anticipates the complexity that intuitive expressions may take. Husserlian theory, with its close description of structural necessities that permeate the lived body, affords an understanding that is *transcendental* rather than *empiricist*.

The evidence acquired by the analysis of the learning episodes that follows aims at testifying the theory, not in an empiricist sense but as essentially and critically transcendental. Husserl's authorisation for more and more *abstract objects* is in contradistinction to Kant (Hintikka, 1972, 1998, 2003) indicates the richness that is already anticipated by the Husserlian theory. And if the cognising subject somehow develops her empirical intuition to further abstract intuitions, the new intuitive objects belong to a new category. Kant lacks oxygen in his contact with unformed and categorically unshaped objects. Experience for Kant starts from the body-as-object and as being objective; the perceptive component that is the ground of our lived experience, the kinesthetic, tactile, auditory, etc. is for Kant "simply received, grasped, and then elevated to the realm of the intellectual" (Radford, 2014, p. 131). Husserl simply justifies and focuses on this real, which he called the *life-world* (Lebenswelt). Husserl's term, *passivity,* hides the activity that takes place in the pre-objective world, which, also constitutes the *void* of many theories in mathematics education,[103] due to their "blindness to intentionality", and their

[103] Starting with constructivism and its numerous explicit or implicit ramifications—for an implicit ramification that lends its theory on intuitions by Fischbein's Kantian tradition, filtered by a socio-cultural methodology and a concomitant data analysis, see Andrà and Santi (2013, PME Proceedings). In the aforementioned research no thematisation or clarification of any link between intuitions and objectification appears, since it is concerned with intuition only when it becomes *shared* within the collective activity; although it addresses a complementary aspect of my research that deals with the constitution that is driven by the 'we-consciousness', it misses *intuition* and *objectification* as it fails to see the crucial links between them.

"blindness" to the *embodied* subjective that synthesises the *ego-pole* and the *object-poles.*

> A thought really transcended by its objects would find them proliferating in its path without ever being able to grasp their relationships to each other, or finding its way through to their truth.... We must define thought in terms of *that strange power which it possesses of being ahead of itself*, of launching itself and being at home everywhere, in a word, in terms of its autonomy. Unless thought itself had put into things what it subsequently finds in them, it would have no hold upon things, would not think of them, and would be an 'illusion of thought'. (Merleau-Ponty, 2002, pp. 431-432, my emphasis)

The Husserlian theories of pre-reflective and reflective lived experience of the cognising subject, informed by Merleau-Ponty's radical reflection and primacy of perception, were put into place, and the data confirmed the theory in expected and unexpected details.

3.1.2 Issues and questions emerging from the choices towards the data collection and analysis – A mapping of the three cases that will be analysed

A core-issue during the exploration of the material was mathematics, as a subject taught and learned, learned and taught, in the same or different cultural environments, where mathematical conceptualisation is a highly rated target. I have been a teacher, formally for nineteen years and informally for a decade more, in a varied teaching environment (private as well as state education) of a European country, and I based my current data on participation and observations of a different environment, the first year that I came to England. Why a different environment? Why the first year? And which mathematics are these that are studied here, since the age, the experience and the motivations vary? In other words, which is the 'invariable' that makes the three cases of this study a significant introduction to an originally new perception on mathematics education, as this study argues that it explores/suggests?

Opening up these issues means starting from the mathematics as:

- The subject of the investigations (the tasks, as they were defined by the teacher)
- Mathematics in the (chosen) context that they were performed, and the particularities that they developed as they were observed, recorded and interviewed.

Given that these two aspects, the *mathematics as a subject-matter* and *the context that they are performed,* are essentially interrelated, the simplest way of bringing these issues to clarity might be to elucidate the nature of the mathematics involved in the analysed tasks,[104] in relation to the results that emerged. I will delimit the knowledge of mathematics to the explicit and implied *intentionalities* of the teacher and the students involved, according to the degree that they influenced the cognitive 'moments' that will be analysed. This of course does not mean forgetting the mathematics but only *bracketing* them in order to conceive the learners' intentional horizons in the studied cognitive 'moments', and to study their paths towards mathematical abstraction. As I was a participant observer, I will delineate my adaptation to the conditions as they appeared to me, but limit my focus on the aspects that had significance for the whole process and the process as a whole, which is not the same thing.

- A key issue that will also be addressed is the 'mathematical neutrality' of the teaching approach, and how it helped particular (intuitive) methods to take hold. A key issue that played a part in the fruition of the research was the reflective character of the sessions, which was mildly introduced and developed towards becoming more systematic way : the students were asked to participate in *reflective breaks,* where they wrote every 20-30 minutes or between two different tasks during the sessions, each time with a given topic; the most regular of these topics was to report on their actions and their position towards their targets.

[104] As well as a broader though necessarily more general picture of the nature of the mathematics involved in the broader context of the sessions.

- Processes such as the teacher prompting the students towards collaborative work, although individual strategies and findings were also well accepted and considered positively.

The freedom to go as they chose to a great extent, as Diana's work (Ch. 3) will manifest, determined the responsibility that the students developed for their creations, in a course that was not always easy, despite (and rather due to) the freedom that it endowed the students for their strategies, as no direction/"guidance" was provided.[105] Most importantly, it gave the answer to what counts as data for this research study, in which tracing mathematical objects that the students devised and their origination was the central theme of investigation; and the answer from my phenomenological perspective was that data is whatever was objectivised for the given task, as standing out from the learner's flow of appearances, in conscious perceptual acts. But this is only the beginning of the story, since this *standing-against* (objectum in Latin and Gegenstand in German) is where conscious acts are intentionally directed towards; while each aforementioned drawing of attention signals a certain sense endowing act that follows, after reflection.

In the broad notion that Husserl means the term, object is defined as "anything at all, e.g., as subject of a true (categorial, affirmative) statement" (1983a, pp. 40-41). Being on the side of Husserl, I followed his methodological framework not simply trying to prove what I already knew; it was about philosophically clearing the way, the prejudices, the taken-for-granted, so that what I would be left with, in Husserl's terms, would be a set of phenomenological descriptions that *stand up to rigorous questioning*.

Ivan's and Diana's cases *are also cases* due to their *complementary* features, due to their *contrasting* features, which is the same thing. One case can be seen as the negation of the other in many of their features (thematisations), originating in their contrasting approaches towards the course and the freedom it allowed to the students: while the freedom was welcomed by Ivan, it was perceived as lack of

[105] See Diana's interview (Appendix C) for the use of "guidance". Diana's case is also informative of the negative aspect that some students expressed.

guidance by Diana; where Ivan was perceiving each task as a new chance to apply his intuitive methods, for Diana it was a case where the other students moved too fast to track them, and where her late realisations were most usually washed out by the other students' ideas, unless they were compatible to their ideas.

But there is something significant that Ivan and Diana share, concerning the feature of satisfaction from the course as a whole, when the sessions were concluded; namely their feeling that the course gave them important re-conceptualisations of mathematics as heuristic forms of pro-active/pre-active experience, as *mathematics-in-the-making*. And that was enough for Diana and Ivan, in order to preserve this course in their memory as a landmark of their studies in order to become teachers of mathematics. And this attitude was shared by the third part of this collection of cases, Mary, who is the representative case of all detected intuitions and objectifications of this study. Mary explored a theme that is considered 'common' for mathematical classrooms[106] brought about issues of reactivation and de-sedimentation of culturally/historically acquired knowledge.

A crucial aspect that will not be neglected is the particular perception of mathematics and of the students' mathematical ideas by the teacher; but this subtle issue will be explored only to the degree that it affected the students' perception of the tasks. In other words the teaching aspect is bracketed but not neglected, as it will return in the discussion of the findings of the three cases. Closely related to that is how this environment facilitated my intended observations during the data collection, and how gradually informing the data theoretically and methodologically brought this study to life, i.e. with its inscription to a Husserlian phenomenological discourse that affords a rich cognitive theory and an according methodology. Testing the theory with the data led to the clarification of the theory rather than any amendment in order to conform to the findings. The results of this happy interplay of theory and data are tracked here, and they are perceived as applicable by mathematics teachers and mathematics education researchers. It is a case where theoretical ideas are shown to be applicable to empirical studies, and the

[106] I.e. the parabola as a curve that is equidistant from a fixed point and a line.

phenomenological methodology to be essentially constituting, while showing how the lived body is constituted. By deploying the aforementioned issues it will become apparent how these three cases that are analysed here were judged as adequate for the study, and how they are adequate cases as such. Main points that support their *double-folded adequacy* and that will be explored in the following sections are…

- That they exemplify in detail a phenomenological methodology, as described in the previous chapter. It all becomes possible due to the exclusive focus on the operative acts that prepare the 'visible' constitutive ones, and make them possible. It is through the successive deepening of the reduction that we are legitimised to anticipate where the unseen operative forces of the lived body are present; to detect them where the traces ought to appear. More precisely, it is due to the theory working with the methodology, detecting what the methodology allowed to be sensed as significant for the theory, that the broader understanding of intuitions and the commencing of objectification that this study argues for is expected to be unfolded. Therefore what counted as data from the operational aspect of the study is what made sense for the theory, as it was revealed by the concordant methodology.
- That they offer different *profiles* of the same *intentional* phenomenon, namely *intuitions* in the Husserlian sense, in service of *objectification*, as they were detected on the students' activities—i.e. the perspective adopted by my research. It is a bond between intuitions and objectification that is thematised for the *first* time in mathematics education research. And the three learning episodes operate further than triangulation, aiming at been perceived as adumbrations of well-known phenomena—intuition, objectification—under a new perspective, namely the objectifying aspect of intuitions. This is precisely what the empirical (analytical) part will be for the theory, namely the exemplification through evidence of abstract Husserlian concepts that allow access to understanding embodied cognitive conscious acts towards the constitution of mathematical objects.

The three learning episodes/students support their *double-folded adequacy* in the sense that they support the argument of the link between objectification and intuition in the Husserlian sense, and they elaborate through their differences and similarities aspects of *intuition in service of objectification,* which would not be visible in the straightforward approach of each single case:

- Each one of the three learning episodes/students is a case, since they cast light on different aspects of *objectification*, including:
  - Different and overlapping types of intuition according to the objects that surfaced and the structure that was at play for each object.
  - Different types of engagement in intuitions (different *states of affairs*) for each student, which determined the intended objects, as intended.
  - Different "manners of giveness" of object-like formations that became objects.
- Examples of *empirical* intuitions and of different kinds of *abstract intuitions*—categorial and intuitions of essences—are given in the three learning episodes/students which, following the order of the presented cases:

  (A) The intuition of essence will be detected twice in the study, firstly in the case of Ivan, where it will be manifested as a *double* shifting of attention to previously neglected evidence and to new aesthetic patterns, due to a symmetry intuition that finally collapsed, both resulting to the three steps of the *synthesis of coincidence* and the concomitant *intuition of essence* (§3.2.3).

  (B) A *categorial* abstract intuition in two stages, starting with an *empirical* intuition that originated in trial-and-improvement, and followed by the *categorial* transformation of questions/answers to an algorithm and then to a flowchart (§§ 3.3.1, 3.3.2). The legitimisation of the results by the teacher had a major impact to the student's attitude towards the course, and her learning and teaching attitudes in general, which could be the theme of a separate study on attitudes and affect. From the perspective of the research, the significance of intuitions for the learner's

mathematical interface and the key part of the teacher's legitimisation of the student's objects are two main features of this case.

(C) An empirical (visual and kinesthetic) intuition that enabled the shifting of attention to the classroom experience, through a re-presentation substantiated in a diagram, and the bird's-eye-view perspective of the classroom through the diagram (§ 3.4.2.2). The last learning episode as a whole (Ch. 4) is already a case from the perspective of *intuitions as objectifying acts,* due to its intuitive diversity and its embodied fullness, fed as it was by the embodied start-up in the classroom. All kinds of intuitions were detected in this case—including the *categorial* abstract intuition in two stages and the *intuition of essence* in three steps—and it is the episode that most consistently studies embodiment and its grounding qualities.

…as they will be unfolded in the coming sections.

3.1.3 Preliminary manifestations - The main points of what is at task for the study - Rationale of the data analysis

The study of the three cases were chosen in order to show the complexity of the learner's lived experience and a phenomenological approach to it, concerned with the *living present* of each prospective teacher of mathematics that is analysed, and attempting to delve as deep as possible into the origins of the learners' achievements.

The phenomenological lens that was used, both in the theory and in the methods/methodology allowed the particular view of the data, which thus became findings. The data analysis itself is the clear evidence (and in some sense the result) of the methodological, theoretical and analytical interplay. The analysis is driven by phenomenological methodological instruments and according theoretical support. What initially appeared as the students—prospective teachers of mathematics in

English schools—"*key ideas*"[107] evolved to the students' *intuitions*, under a very particular Husserlian approach that draws on pre-Cartesian understandings of the term (Hintikka, 2003). Merleau-Ponty's advancement of Husserlian key ideas allowed the theoretical and methodological lenses to approach the lived-experience of the learners from a novel Husserlian perspective. The aforementioned perspective casted light to the structure of empirical and abstract intuitions, and it was enriched empirically by bringing evidence to and for the theory. The generality and originality of the three cases are expected due to evidence that the analysed episodes concern practices that may be *common* although *neglected* in mathematics classrooms. It is for the latter reason that this research study calls for awareness of

- The learning praxis, as a phenomenon where *intuitions* in the Husserlian sense are key features of *objectification* (and thus of knowledge production), allowing access to the objectification process from the intuitive operations in the pre-reflective stage to the intuitive acts in the reflective stage.
- The teaching praxis, primarily as an event that could allow learning to take place rather than being "the procurement of useful information" (Heidegger, 2004, p. 15). Teaching is expected to let learning take place by nurturing and provoking intuitive happenings and being ready to rip the fruits, by legitimising mathematically what is intuitively grasped, rather than canceling what is outside the scope of the curriculum in the narrow, classroom sense. Such a style of teaching would go against the promotion of the prestige of the learner, or of any authority, including the self's own teacher persona.[108] The legislated body of mathematics itself, allows a range of alternative approaches that transcend the legislated curriculum, in favour of

[107] The term "key ideas" will not appear often in the study, since they will be soon replaced by the key ideas' constituent parts, namely various kinds of intuitions. The term appears again in the description of the *natural attitude*, which is the naïve belief in the appearance of things as already there, and in that case it amounts to the 'blindness' to the *intentional* direction of constituted objectivities by constituting subjectivities and their collective communities (the 'we-intentionality').

[108] "persona: The Latin word for person, q.v., used in certain phrases: b.4.b In Jungian psychology, the set of attitudes adopted by an individual to fit himself for the social role which he sees as his; *the personality an individual presents to the world*;" (Oxford English dictionary, digital edition, emphasis added).

the students' intuitions. And these intuitions anticipate the teacher as the introducer of the learners' lived experience to the formal mathematical domain. The introduction of the learner's intuitions to formal mathematical expressions amounts to the inclusion of the learner's experiences, through their transformation to genuinely mathematical experiences; thus transforming teaching and learning to teaching and learning praxis respectively.

This was the pathway chosen in this study in order to investigate closely the students' deep engagement with mathematics, the abstract queen of the sciences, by bringing mathematical abstraction down to its origins in lived experience. This path is mediated for the *learner* to *mathematics* through the *teacher* who understands, translates, promotes, communicates and enhances the intuitions he receives and unearths in the classroom.

3.1.4 Ethical issues

The ethical implications of my academic activities include the consideration of the data that was collected, although there was not sensitive data with particular regard to matters such as age, colour, race/ethnicity, nationality, disablement, religion, sex, gender, sexual orientation, personal medical records and political beliefs. There was not any relationship with the participants, other than that required. The arrangements for the security of data, participants and confidentiality include using pseudonyms throughout the study, as well as in my published material (ESM, RME) and in all presentations of my research in conferences (BSRLM, PME). I have also made sure that any original data sources (c.f. Part 2, §2.3) are only in my possession and nobody will have access to it. Concerning the written work of the students in particular, nobody except the teacher of the class will have access to it. In other words, what I am using as data is only for my research and for no other purpose, and the anonymity of participants is ensured.

No payments were made to the participants for participating in this research. And no other external rewards, such as increased marks were given to the students. No special indemnification arrangements were required. Due to the nature of

research, in which no harm could come to the participants, and no external rewards were offered there was no external agent needed; and there was no third party involved in the University's activities. The size of sample proposed was not an issue for my study, due to the qualitative character of my research. My research did not cause any distress to the participants; on the contrary, it might have been beneficial for them. My purpose was open for the participants right from the beginning; the students were aware of the target of my research, expressed as 'the better understanding of their mathematical investigations', and I was not deceiving the participants in any way. Explicit verbal consent was given by all the participants, and the students were aware that they could withdraw from this research project at any time. There were no participants who might be unable to assess the implications of the proposed work, and there is no potential risk to the University, in order to be outweighed by the value of the academic activity. The risk to the University in terms of external (and internal) perceptions of the worthiness of the work has been assessed and is deemed acceptable; the positive assessment of my research concerns the publication of a major part of the study to peer-reviewed journals (ESM, RME) and conferences (PME, BSRLM). Arrangements are in place which safeguard the interests of the researcher(s) being supervised in pursuit of the academic activity objectives, and permission was obtained for the special arrangements that have been made for the security of related documentation and artefacts.

In summary, all ethical considerations were taken into account in this research. The aforementioned response to the list of ethical principles covers all issues related to my research project, and my response to it indicates how I complied with all the ethics principles that are relevant to my study.

CHAPTER 2. Ivan – A mature intuitive approach on mathematics that was released due to the special character of the NoM course

3.2.1 Introduction to Ivan's case study – Rationale of the research – Methodological considerations

This is the case study of Ivan and his decisive intuition, as my research attempts to shed light on the objects that the student constituted in order to manage his mathematical experience. We will follow the emergence of such an object, whose appearance became possible due to a particular kind of intuition that the Husserlian theory anticipates, the *intuition of essence*. In order to do so I will employ the Husserlian frame that intuitions are seen through, and their main properties/features that make them detectable in everyday learning praxis, such as the one that will be analysed here. I will focus on the findings, and exemplify the theory through the structures that emerged from the analysis of the data, after the application of the Husserlian theoretical and methodological frames.

Some allusions concerning the distinctions between the theory proposed here and Cartesian and Kantian approaches to cognition will be given, after certain evidence will be traced in the analysis of the learning episode. But the main theme will be to exemplify the theory with concrete examples, which were made possible due to participation, observation, three interviews (two short and a lengthy one) and copies of the students' coursework. The students' warm and open cooperation and the teacher's generous access to all these sources made this work possible. The teacher's nonintervention approach to teaching allowed bracketing his involvement and focusing on the students' own ideas and strategies;[109] The teacher's particular method of abstaining from any guidance during the sessions allowed to concentrate on the students' own actions; in phenomenological language, the teacher's nonintervention strategy allowed *bracketing* his involvement and facilitated the *reduction* to the students' own ideas and strategies. In general terms, my study

[109] The shift from the singular (the particular student) to the plural (the students of the course) is due to the applicability of this statement to the other two cases of this study, as we will see in the following sections/chapters.

employs the aforementioned phenomenological methods of bracketing and reduction in order to explore *intuitions as critical objectifying acts* (Zagorianakos & Shvarts, 2015).

The learning episode concerns prospective teachers of mathematics and it starts in the classroom, where the students worked in groups of 3, 4 or 5 people of their own choice. I will focus on Ivan and follow his actions in his group, the exchange of ideas, and the manner in which he used them when he went home. And I will do that by bracketing anything but the student's intentions and the intended objects (his empty indications and how they were fulfilled, as well as the objects that were constituted due to the fulfillment).

It is a learning episode where the student followed his instinct throughout his investigation, although exploring ideas coming from another member of his group. And the teacher's withdrawal from any sort of instruction or guidance helped immensely to focus on the relation mentioned earlier, between the student's intensions and the intended objects. Bearing this in mind we will realise that the tracing of the structure that emerged and its recognition (identification) as an *intuition of essence* in the following sections has become possible due to the particular manner of looking at the student's experience. This structure will be analysed as it appeared in three steps, namely the step of the *starting example*, the step of the *variation of the starting example*—when the starting example is modelled on another example—and the step of the *synthesis of coincidence*, when the two examples lead to the intuition of generality.

Although from the point of view of this research every learning episode may be transformed to a case, under the appropriate phenomenological lens, Ivan's learning episode is primarily a tribute to a student whose intuitive tendencies had already acquired a seemingly 'natural' sense. As it was the case with all sessions that I participated anywhere near Ivan, he was following his rules, he was incorporating ideas easily, and in this 'flexibly self-determined' manner he was deeply concerned with the results he wanted to achieve and his ways of getting to them. Being very intuitive and innovative he was also examining me for any kind of

information he might get.

And it did not take me long to discover (revealed also by his lengthy interview) that under this 'natural' intuitivity there was a long mathematical *practice* involved, which was bringing together his mathematical competency and his mathematical interface in general, with an idiosyncratic tendency to aesthetic forms, as the following learning episode will only exemplify.[110] Ivan's genuine interest in mathematics also led him to reflect on his mathematical experience, as his lengthy interview will show.[111] My exceptional cooperation with him for more than 3 years was invaluable for the completion of this study.

The NoM course was Ivan's favourite and it remained so up to the end of his studies, as he has had expressed it to me for a few times. It was the only course that had allowed him to constitute freely his conceptualisations of the given tasks and put his mathematical ideas into motion.

3.2.2 The course and the 'New York cop' task

The nature of the course allowed many initiatives to every student, concerning the directions they might take, encouraging group and inter-group collaboration. The 13 students (prospective teachers) had the opportunity to choose 3 or 4 of their favourite activities in order to further develop and present them in a written form at the end of the course, in order to get their marks for their overall performance. During most of the activities there were *'reflective breaks',* where the students wrote what they were doing. The students were asked to add this *'reflective practice'* to the writing of their aforementioned coursework, delivered at the end of the course.

- The minimised instructional character,
- the open-endedness of the activities

[110] For further familiarisation to Ivan's tendency to aesthetic forms one may refer to the appendix, for Ivan's chosen activity—that each student produced at the end of the term—as well as his approach to the 'doubling modulo' activity, which is analysed in Diana's case. Even more evidence can be found in Ivan's coursework and his interview (also in the appendix section).

[111] Apart from the extracts used in the following analysis the whole interview can be accessed in Appendix B, Paragraph B).

- and the reflective feature of the tasks and the coursework

facilitated the students' intuitive approaches, which became more explicit than in many 'typical' conventional or even exploratory teaching frames.

The teacher adopted a merely operational instructional disposition during the sessions, which enhanced the effect of the reflections and the individual/group independent methodologies and practices (individual and collective). One of the students that took seriously the teacher's call for taking the initiative and their own ways of tackling the problems/tasks, was Ivan. He acknowledges it clearly in his lengthy interview, when he I was invited to the sessions by the teacher and I was given access to their coursework, in order to collect data for my research, and the students gave me permission to participate, observe and record their activities. I'll acknowledge my thanks properly, at the conclusive section, and at the corresponding sections to the according analysed cases, in the analysis chapter. All the students' permission to participate and record the activities was genuine, and they were very open, throughout the sessions.

3.2.2.1 The first part of the session

The task that will be analysed was called 'New York cop', and it lasted two and a half hours, while there was a half hour break after one and a half hours. My research concerns the activity of the group consisted of Ivan, Carry and Donald (pseudonyms). The teacher described the task as follows:

> T: Here we've got a small precinct [the teacher drew one small square on the board] and basically you've got a policeman standing there, [he added a 'cop' at the bottom right corner] who is able to see *two blocks* along. [he added three more squares –Figure 1] Now, this isn't New York in the old days where you had crime; *how many policemen would you need* to see along *every street,* in that configuration? [Donald says 3] So where would you put your 2 [cops] Donald?
>
> D: The way it is at the moment I would put one at the opposite corner, one in the middle and one at the opposite corner.

T: Is that ok? So 3 policemen could actually cover every single street; now these policemen are relatively *short-sighted policemen,* who can only see *two blocks*. So the question is, as the precincts get bigger how does the number of policemen you need increase. Ok?

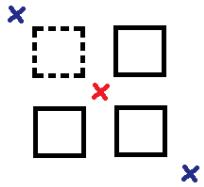


Figure 1. The teacher's first drawing; the dotted square and the cop in the middle were drawn first, and the other 2 cops were suggested by Donald after the teacher drew the other three squares.

The students were divided into 4 groups of 3 students of their own choice (one student did not come that day). I recorded the activity of the group consisted of Ivan, Carry and Donald, sitting next to their table and observing their activities most of the time, and I did not intervene during their investigation. I took short interviews from Ivan and Carry at the beginning of the second part, after obtaining the students' permission. At the end of the session the students gave me the notes that they kept during the session, which I photocopied before I returned them to the students.

The task was targeted in pursuing generalisations related to configurations of the 'short-sighted cops' that would give the minimum number of cops for each grid, and eventually consistent patterns that would allow them to find formulas. But, as the teacher put it half way through the session,

> What I think we are aiming at here and I think it's *extremely* important is rather than just being about getting the formula, is actually being *aware* of the *systematicity* of the ways that you'll locate it, that you're actually learning to *position* the policemen in a *consistent way,* because you've got an *intelligence* about *moving* around the *space,* and *seeing* how you can *cover it.* (italicised words are due to teacher's emphasis)

Ivan and Donald started by exploring square precincts, while Carry was investigating 2×n precincts. Ivan and Carry developed radically different methods, since Carry was focusing on more local generalisations (for the 2×n precincts and later for the square precincts), not necessarily related to formulas (as her interview revealed), while Ivan was aiming at the broadest generalisations possible and the formulas that could pin them down, as his actions up to the end of the classroom and home investigation revealed. Donald did not manifest any particular methods throughout the session, nor did he influence either Carry or Ivan's investigations. Nine minutes after the students started exploring the task Ivan introduced the 'inside cops' idea and his strategy—which became consistent later in the task—emphasised the prominence of how the cops should be placed inside the precincts. Carry, on the other hand, prioritised the placement of the 'outside cops', that is the cops around the precincts, since she believed that "you can always work out the minimum number of cops you'll need on the outside because of how far they can see" (interview extract). As a result of her strategy, Carry was starting her positioning of the outside cops by consistently putting the first cop at the top left corner (second drawing in Figure 2), which is what Donald also did with his square precincts, and both following the initial drawing that the teacher did on the board (Figure 1), which remained drawn there throughout the session. Ivan started by doing the same, but his 'inside cops' strategy eventually led him to a method that was based on starting with 2 cops on the outside, which both had a 2 blocks distance from the top left corner of his drawings (Figure 2). The different strategy had major implications for the methodical coherence of the group, since Ivan and Carry *developed different strategies throughout the session,* while Donald's treatment was a combination of Ivan and Carry's methods.

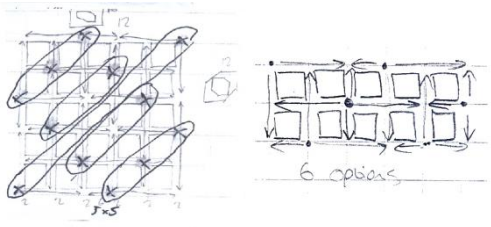


Figure 2. Ivan and Carry's different strategies for the 'outside cops'

Carry came to a conclusion about her 2×n precincts, before the end of the first half of the session: "as you add a column at each time, you will need one cop more than that column, to cover all the streets" (interview extract). She did not seek for a formula in order to describe this result, since "what I've been trying to find is not a formula, I'm trying to find a common way of increasing the cops each time, and seeing from that pattern something that matters" (interview extract).[112] As she was trying to find a "pattern", i.e. "a common way of increasing the cops each time" she did not explore different cases where the number of rows is fixed, such as 3×n, but she explored the square grids; and that is when her different strategy (outside cops) to Ivan's (inside cops) came out. The two students did not come to a settlement of their dispute, but from her engagement with the 2×n grids she brought to the group the 'vertical streets' concept, that is the amount of cops for each vertical line of the grid, which Ivan adopted.[113]

Ivan was also struggling to find configurations that would give the minimum number of cops for his grids, which would lead him to generalisations. He was keeping an account of the cops in each vertical street, the amount of the inside cops (a number under each grid at Figure 3) and the shape of the inside and the outside cops (at the top right of each grid at Figure 3), while checking symmetry in his drawings (Figure 3). At his interview that took place at the beginning of the 2nd part of the session,[114] he mentioned the *diagonal symmetry* that he noticed for the 4×4, 5×5 grids (Figure 3) and his prediction that it might "go on to six or so on". Being preoccupied with symmetry, Ivan was puzzled when he discovered an asymmetrical configuration that gave a smaller number of cops for the 3×3 grid (Figure 5). Right at the end of the first part of the session, just before the break, he mentioned to Carry that "[t]hey [the patterns] might have to be *asymmetrical".*

[112] Carry's short interview can be accessed in Appendix B, Paragraph 2).
[113] See the numbers that Ivan has written under his grid in Figure 6.
[114] Ivan's short interview can be accessed in the Appendix B, Paragraph 1).

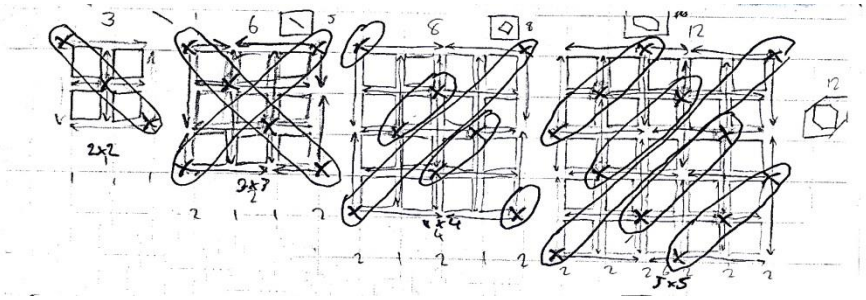


Figure 3. Ivan's initial drawings of squared precincts, from his notes during the 1st part of the session

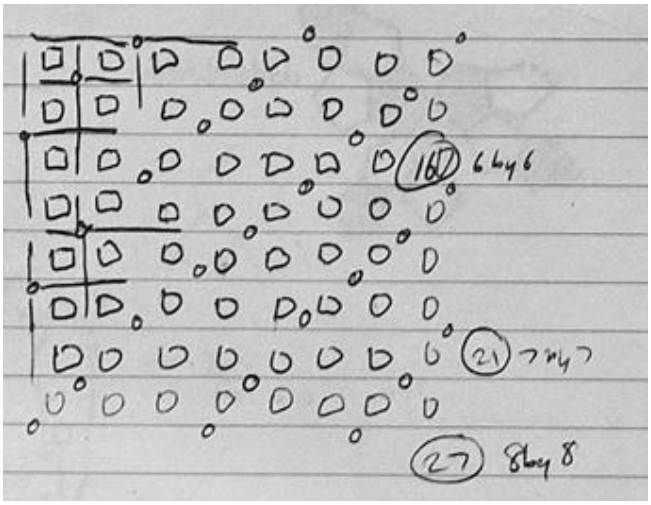


Figure 4. Ivan's drawing during the 2nd part of the session

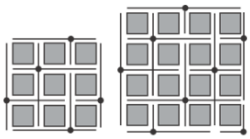


Figure 5. Ivan's asymmetrical drawings for the 3x3, 4x4 grids

3.2.2.2 The second part of the session

Towards the end of the activity the students wrote a summary of their findings. Carry did not systematically explore cases where the rows are fixed and different than 2, and she did not manage to find patterns for the cops in the squared precincts. Her findings were limited to the relation she had found during the first part of the activity, related to the $2\times n$ grids, still without using any algebraic notation; and the only reference to the square grids was the following:

With regard to squared precincts I know how to work out the min number of cops around the outside of the precinct in relation to the number of columns.

Perimetre of 4 → 2 outside cops

of 8, 12 → 4

16, 20 → 6

Donald merely summarised some of the findings of his peers (e.g. Carry's 2×n finding and Ivan's diagonals), adding that "I think I need to do a lot more work on this problem to find an answer". Both Carry and Donald did not continue their investigation at home.

Ivan wrote in his summary that "I feel if I carry on I will eventually come to a solution but I don't know how long it will take"; and he continued his investigation at home. But where did his feeling that he would come to a solution derive from? His notes during the second part of the session are quite revealing: Ivan found another asymmetrical pattern for the 4×4 grid (Figure 5)—using the same pattern as the asymmetrical 3×3 grid he had already found (Figure 5)—that gave the same (minimum) amount of cops as the symmetrical one at Figure 3. He drew the symmetrical and the asymmetrical 4×4 grids next to each other and underneath he drew the 5×5 grid, in which he used the same pattern; but the 5×5 grid also had diagonal symmetry (Figure 3). He decided to pursue the asymmetrical 3×3 pattern for more grids, and a *diagonal pattern* emerged (Figure 4). But, as he explained in an e-mail that he later sent to me, "I had the pattern down before the end of the lesson, but hadn't fully understood at that point the significance of this".

The richness of different options and ideas that either he had found or Carry had suggested, and his preoccupation with symmetry did not allow him to further pursue his attention to *asymmetrical* patterns that he had expressed at the end of the first part of the session, but the discovery of the 4×4 grid and of the *diagonal pattern* had opened a new field of exploration for him, one that he took up as soon as he returned to this task at home. Therefore the legitimate claim here is that the aforementioned activity towards the end of his classroom exploration afforded him—

through his lived body— with the feeling that he expressed in his last classroom reflection. And it is the reason that his last reflection is registered as expressing an intuition (that a solution is within his grasp), which we will see how it was fulfilled during his investigation at home.

3.2.2.3 Ivan's investigation at home

At home, Ivan put next to each other his symmetrical and asymmetrical drawings for the 3×3 and 4×4 grids. He drew the 3×3, 4×4, 5×5, 6×6, 7×7 and 8×8 grids using the same 'asymmetrical' pattern for all these grids. As he wrote in his coursework:

> I decided to review it [i.e. the table] to see if there were any patterns forming. At first it seemed like there wasn't any logical way forward so I decided to approach this task from a different way. I decided to concentrate on grids with the same height. The first group of grid sizes I looked at were ones with height 3 but this time I was also looking for a pattern I could use to get the optimum amount [of cops].

His first step forward was that he was testing a unified way of dealing with the precincts and the first time that he "decided to concentrate on grids with the same height", i.e. different than the square grids. His second step forward was that he no longer identified *pattern* with *symmetry,* since he started with the asymmetrical 3×3 grid (Figure 5) in both the square and the $3 \times n$ grids. Ivan described in his coursework what happened after he drew the 3×3, 3×4, 3×5, 3×6, 3×7 and 3×8 grids (Figure 6):

> These grids were a *break through moment* for me for two reasons. The first reason was because there was a logical order to the number of cops per column, it is a sequence of 1, 1, 2, 1, 1, 2, 1, 1, 2. The second break through was a pattern of cops that was the optimal solution, it seemed the best way to arrange them was in diagonals. (italics added)

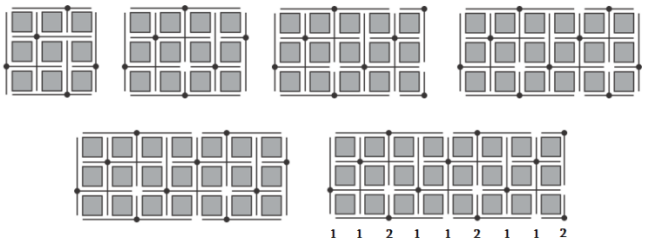


Figure 6. Ivan's 3x3, 3x4, …, 3x8 grids, copied from his coursework

3.2.2.4 Linking Ivan's case to the theory (theoretical analysis of Ivan's case)

The "break through *moment"* indicates the *immediacy* that intuition operates. In the first place, what Ivan depicts here is a form of "perception [which] is characterised as bringing its object to an originary kind of presentation", "[i]n contrast to various kinds of presentifying (*vergegenwärtigende*) acts, such as recollection, fantasy, or empathy" (Zahavi, 2003, p. 162). According to Husserl the element of *immediacy,* accompanied with the *intentionality* to find a pattern for the optimal number of cops that is not necessarily symmetrical, and the *feeling of certainty* after he saw that the pattern was also working for the 3×n grids, are what make this perception an intuition.

But there is a deeper reading of the "break through moment" that Ivan expressed in the summary of his activity, which Husserl's theory allows us to realise, from the point of view of Ivan's intentional origins: Ivan resisted exploring any other case apart from the square grids in the classroom, as he prioritised finding an *appropriate pattern* for the optimal amount of cops. Even at home, Ivan did not explore the 3×n case before he had a hint that he had found such a pattern, one that could suit every particular case ("a pattern of cops that was the optimal solution"). His *feeling of certainty* that "the best way to arrange them was in diagonals" meant that his intention was finally fulfilled by an object, namely the *diagonal pattern.*

This is how, according to Husserl, we can detect when an intuition takes place: by a new *object* that *fulfils* an *intention,* in a *'moment'* (i.e. with *immediacy)*. The object is brought to the surface of consciousness as it is separated from the subject's world-as-lived. And it is in this sense that the object of intuition is not a

construction, since it *becomes known* as it is separated from the subject's reality, in the "overlap of my consciousness and reality" (Hintikka 1995, p. 82). There, on that piece of paper, the object was suddenly detected as such; it became a *figure* on the *ground* of the paper and the drawn grids,[115] when Ivan separated it as the *asymmetrical pattern* that could be the *actual* starting point of his investigation of particular cases such as the 3×n one. And, most importantly, this research suggests that Ivan's investigation reached its vocation only when Ivan had the *feeling* of certainty that his original intention was fulfilled, due to his *intuition* that the diagonal pattern was the essential pattern that he could use *for every grid.*

3.2.3 Ivan's intuition of essence – a detailed account back to the intentional origins – an X-ray view of the startup of objectification

Let's take a closer look at Ivan's pursuit of the appropriate pattern for every grid, throughout his investigation: the diagonal pattern had already appeared in the 3×3 grid (Figure 5) during the first part of the session, but Ivan ignored it since it was asymmetrical. It appeared again in the case of the 5×5 grid but this time the pattern was symmetrical and he did not see the connection between the 5×5, 3×3 grids; thus he grouped the 5×5 grid with the symmetrical 2×2, 3×3, 4×4 grids (Figure 3). It appeared again in the case of the 4×4 grid during the second part of the session, and at that time he saw the connection between the 4×4, 5×5 grids, and he eventually found the diagonal pattern (Figure 4); but he still could not understand "the significance of this". Ivan wrote in his summary that he had the feeling that if he would carry on he would eventually come to a solution, and he decided to continue his investigation at home; it was this 'gut feeling' that made him persist and continue his exploration at home.

When he reassessed his classroom activity he *isolated* the asymmetrical 3×3, 4×4 grids and he put them next to their corresponding symmetrical ones, thus realising

[115] The way that the 'figure on the ground' is meant here derives its use from Merleau-Ponty's Phenomenology of Perception (2002, p. 4)

that they are more effective.[116] It was an act of *attention* to the *'asymmetrical' diagonal pattern* and according to Merleau-Ponty (2002, p. 33), attention "first of all presupposes a transformation of the mental field, a new way for consciousness to be present to its objects". Ivan used the 3×3 grid as the starting point for the square and the 3×n grids. His *feeling* of certainty that the 'asymmetrical' diagonal pattern could be effective for *all* the grids was enhanced when he saw that this pattern worked in the case of the square grids (3×3, 4×4,…, 8×8) and the 3×n grids (3×3, 3×4, 3×5,…, 3×8). Ivan needed a mathematical verification that this pattern was significant and he found it for the cops in the vertical streets of the 3×n grids (1, 1, 2, 1, 1, 2, 1, 1, 2 – Figure 6). This simple arithmetic sequence or "logical order to the number of cops per column" as he called it, offered Ivan *mathematical* legitimacy, in order to establish the 'asymmetrical' diagonal pattern as the appropriate one, in his quest for a pattern that could fit every grid. In Husserl's theory intuition is the act of perception that takes place precisely the moment that an empirical or abstract object fulfils a particular intention. The second test that Ivan put the diagonal pattern into (i.e. the 3×n grids) allowed what Lohmar (2010, p. 79) calls the "*synthesis of coincidence"*, which is the core of Husserl's *intuition of essences* that we saw it being performed in three steps, as Lohmar indicates:

- The *starting example* of the 3×3, 4×4, …, 8×8 grids (ibid., p. 78), where the pattern that appeared in the 3×3, 4×4 grids is applied 'extensively' (specifically), being recognised as *worthy for exploration.* The 3×3, 4×4 grids where the pattern is spotted for the first time is what Lohmar calls the *object of experience* (ibid., p. 83), and the pattern is already there but not yet objectified, i.e. not yet separated by Ivan, from his lived-experience. In concrete terms, *he cannot yet see it as an object,* as an autonomous knot of features that satisfy the terms of the task at hand. But as his attention is drawn into it he does see that there is something there, something worthy to investigate further.

[116] This time he chose a different symmetrical configuration of cops than the one shown at Figure 3, which gave 9 cops instead of the 8 cops that the asymmetrical 4×4 grid gave.

- The *variation of the starting example* (ibid., p. 78), when Ivan modelled the pattern he used for the square grids on 3×n grids. The reflection of the effectiveness of the 3×3, 4×4 grids' pattern on the 3×n grids, after it was applied on the square grids, became decisive for the 'asymmetrical' *diagonal pattern* of the next and final step, since it paved the generality of its applicability as a pattern.
- The *synthesis of coincidence* was realised at the moment of the *immediate* recognition of the 'asymmetrical' diagonal pattern as "the best way to arrange" the cops, as the generalisable pattern that he was intending from the start. The view of the intentional origins of the pattern appearing in the 3×3, 4×4 grids, namely of the intention that synthesised them as a pattern that could give the best results *in general,* becomes possible due to my Husserlian approach to (lived) experience, to learning as learning *praxis*, to teaching as *invitation* to common learning. In terms of the theory, the intuition of essence takes place at this moment, the moment of the *synthesis of coincidence*, when an essence emerges out of the successful mirroring of the starting example to its variation; and in this case it was the essence of *configuring the infinite space,* by using one pattern while complying with the terms of the task at hand—the short-sighted policemen. The pattern appearing in the 3×3, 4×4 grids was stripped of its origination, and it was seen (abstracted) as the "pattern of cops that was the optimal solution" for *all* grids. This moment brings to the surface an object of a new category, and in this sense the *intuition of essence* is a *categorial* intuition, as Husserl's theory anticipates (Lohmar, 2010; Hintikka, 2003) Additionally, the *arithmetic pattern* of the vertical line cops testified the mathematical legitimacy of the shape pattern, and it would soon be recognised and cashed in as a source of mathematical abstraction (figures 7, 8, 9 in Part3 §3.2.3.2).

Through these three steps, the *asymmetrical diagonal configuration* of the cops that appeared at the 3×3, 4×4 grids was recognised by Ivan as the 'leading pattern'. The first two steps were preparatory, while the 3rd step was the conclusion and the core of the intuition. The "logical order to the number of cops per column" (i.e. the

"sequence of 1, 1, 2, 1, 1, 2, 1, 1, 2") confirmed the embodied (visual and kinesthetic)[117] intuitive recognition of the 'asymmetrical' diagonal configuration of the cops as the "optimal solution", and it attributed *intersubjectivity* to what his operative intentionality had recognised as the *diagonal pattern configuration.* In other words, the intersubjectivity achieved by the arithmetic pattern made his realisation communicable, as Poincare (2007, p. 348) explains:

> Sensations are intransmissible, or rather all that is pure quality in them is intransmissible and forever impenetrable. But it is not the same with relations between these sensations. From this point of view all that is objective is devoid of all qualities and is only pure relation.

Moreover, Ivan arrived at the arithmetic pattern due to his persistence in finding a pattern for every grid, and I had not initially confirmed to which extent his recognition of the 3×3, 4×4 grids' pattern as the one he was looking for was decisive in gaining the arithmetic pattern itself. What I was certain of was that Ivan's gut feeling that there is a pattern for every grid fueled his investigation from his quest for symmetry, up to the exploration of the asymmetrical pattern. And that the arithmetic pattern provided the shape pattern with apodictic evidence (formally verified) that the diagonal pattern was appropriate for generalised use.

The 'synthesis of coincidence', as it is exemplified in Ivan's *intuition of essences* concerns *the qualitative transformation of the particular, out of the awareness of generality.* The particular is in this case the 3×3, 4×4 grids' pattern (Figure 5). The generality concerns the transformation of the *object of experience,* i.e. the cops' configuration of the 3×3, 4×4 grids' pattern, to the diagonal pattern *that will be used for all grids,* which was what Ivan intended to ultimately find. The teacher had repeatedly asked the students to realise how they make sense of space

[117] I will not explore here the pre-reflective kinesthetic aspect of this intuition, nor exemplify the kinesthetic (sensorimotor) substratum of the (apparent) visual aspect of Ivan's intuition. It may suffice to hint that the creation of subjective sense of space through the drawing of these grids was a premise for the visual apperception (i.e. the perception of the pattern as a solution for all the cases that Ivan had not yet drawn –for Husserl, seeing *another living body* as a subject or cogito is a typical example of an *apperception* (Moran & Cohen, 2012, pp. 39-40; cf. Glossary/Index).

in this task: "What you are trying to do is actually *understand* the *space prior* to fixing a *formula"*; and Ivan's intuition that brought the *diagonal pattern* to the surface of his consciousness—thus objectifying it—was a sense-bestowing act that allowed him to achieve his own understanding of the grids' space; it was the first essential step towards unlocking the task and arriving at 9 formulas (Figure 9 in Part 3 § 3.2.3.2), that gave the optimal number of cops for every a×n grid.

We also notice here as in an X-ray the interplay of *intentionality*, *immediacy* and the *feeling of certainty* in all three steps of the intuition of essence: The second step of the *variation of the starting example* has the effect of enhancing the *feeling of certainty* that the pattern in the 3×3, 4×4 grids is the 'right' one, although the feeling came to its peak only in the accomplishment of the third step. *Intentionality* is performed throughout the three steps, as Ivan checked that the amount of cops is the optimal one in the square grids and in the $3 \times n$ grids, as he was extending the validity of the square to the orthogonal grids; it is clearly expressed when he wrote "I decided to concentrate on grids with the same height". *Immediacy,* which is the critical identifying property of intuition, is performed at the third step. It is the "*break through moment"* that Ivan writes about, that this particular *diagonal* pattern of cops is generalisable, as he recognised it as the optimal solution; it is the moment that Ivan grasped with a *feeling of certainty* the idea that *this* pattern is the object "that it filled the *space"* in "the best way to arrange" the cops.

The intuition of the "shape pattern", of the 'asymmetrical' diagonal pattern as the optimal solution, *came first;* the lived body signaled first that this is the figure to work with. It was an embodied operation that started during the first two steps of his intuition, when he persisted in using the same pattern in $3 \times n$ orthogonal grids although "there wasn't a logical way forward" in the square grids, although he could not detect in the square grids any regularity that would allow him to gain further evidence for the *mathematisation* of the cops' configurations. It was after the lengthy interview that I testified that the arithmetic pattern 1, 1, 2, 1, 1, 2, 1, 1, 2 in the vertical street cops came only *after* Ivan was intuitively convinced that the pattern serves its purpose in keeping the minimum amount of cops not only in the starting and in the modelled example, but also for *all* grids. It was then that the

arithmetic pattern sealed the intuition as mathematisable and made him "decide things", which led to his formulas. Ivan makes it clear in his interview:

> R: So the "breakthrough moment" that you mention in your coursework is the moment that you found the arithmetic pattern, 1, 1, 2, 1, 1, 2?
> I: Well, the shape pattern came first, that was the main breakthrough because I could see that it worked, that I could apply that; then, looking at the verticals is what made the breakthrough for me to decide things. So, it was first the diagram sight, … that the major thing is that it filled the *space*, you could *see* everything, and that's what, what's the task really.
> R: What did you need in order to realise the *"significance"* of this?
> I: [I needed to] sit down and write it, that's it! [laugh] I don't know, I don't know what made it purely significant. I knew it was significant because, no matter how much I extended it the pattern would go on! And, a pattern is generally what you look for in maths, it's how you determine certain things are going to happen over and over again. Looking for constants in nature as well if you… but that's a different side of things. Looking for a pattern was the first part, then I think I went "oh! That goes 3, 3, 3, and *I can do it formal from that!* That's what I think, except I did it with twos, 2, 1, 2, 1… [he remembers what the pattern was] 1, 1, 2, 1, 1, 2, I think that's what the pattern form was.

The arithmetic pattern sealed the intuition as mathematisable since it triggered him to "do it formal from that". But the pattern that would "determine certain things are going to happen over and over again", the pattern that "no matter how much I [Ivan] extended it the pattern would go on" is the "shape figure", the 'asymmetrical' diagonal pattern, and it should not be confused with the arithmetic pattern. He was convinced that the pattern worked after he intuitively synthesised a coincidence out of the two cases (the square and the orthogonal grids) to which he applied it; the synthesis of coincidence of the two cases brought out an *essence* concerning the diagonal pattern, it was a sort of miniature proof by example that took place, in an instance. Two cases, the starting example and the variation/modeling of the starting example created a sort of *stereoscopic vision*

where the new object was born, i.e. seen for the first time. As the teacher had told the students during the session:

> What we are trying to do is actually develop *sensitivity* and *intelligence* to *moving* about the *space,* getting a *sense* of *where* you locate the place in a *consistent* way, so that you'll get a *feeling* of how this space *grows,* of how the supply of policemen *increases* in *relation* to the *different ways* in which the precinct *expands.* (italicised words indicate his emphasis)

And Ivan actualised just that, in a unified way with his intuition of the diagonal pattern, stating it clearly when he says that "the major thing is that it filled the *space*, you could *see* everything, and that's what, what's the task really".

3.2.3.1 Reflections on Ivan's intuition of essence and two features anticipated by the theory – Response to Cartesian and Kantian/constructivist views

The Husserlian approach to intuition that I exemplify here is also a strong indication of the distinction between the Husserlian intuitions and Descartes' *cogito* insight or the Kantian perception of *experience*. In Cartesian and Kantian based theoretical frames knowledge originates (respectively) from the *cogito* (the thinking mind, the 'I think'), and from the process of *understanding* as separated from *sensibility* and already categorically operating. But this constructivist approach to cognition simply will not do for Husserl, who sees the problem of the (naïve) empiricist notion of representations entangled in the schism of sensibility and understanding. Mature Husserl clearly points to the lack of an intuitive exhibiting method as the reason for Kant's mythical constructions (1970a, §30, cf. Husserl's marginal comment in p. 116), showing how experience in the Kantian sense is *ready-made* experience, where Meno's paradox cannot acquire any legitimate cognitive answer. Husserl brings about similar questions when he states…

> If Kant, on the other hand, in the questions he posed and in his regressive method, also naturally makes use of the pregiven world but at the same time constructs a transcendental subjectivity through whose concealed transcendental functions, with unswerving necessity, the world of experience

> is formed, he runs into the difficulty that a particular quality of the human soul (which itself belongs to the world and is thus presupposed with it) is supposed to accomplish and to have already accomplished a formative process which shapes this whole world. But as soon as we distinguish this transcendental subjectivity from the soul, we get involved in something incomprehensibly mythical. (p. 118)

The *soul* here is the *living body,* participating in the paradoxical double feature of the ego according to Husserl, the coexistence of the *subject-for-the-world* and the *object-in-the-world* (Husserl, 1970a, p. 180, Carr, 1999). The body is the field for this tension, since it is where this double feature is expressed in the body's *Leib* and *Kőrper* nature, namely the body-subject or *living body,* and the body-as-object. [118]

Husserl's introduction of the living body and its powers brings into play the *embodied* consciousness, which is crucial in understanding Ivan's building up of his feeling of certainty, up to the intuitive moment of the third step. This building up of certainty for the effectiveness of the pattern was not primarily a thinking process or a separation of already existing or imposed categories from experience, by a categorically afforded mind; rather, he separated what he himself allowed to emerge as a result of his methodology, shaped around the quest for a unifying pattern, and of his openness to visual patterns. It was not a result of acts of thought, but rather a process of 'I can' and a result of 'I see'. The objects that he drew, the different grids with different or similar patterns, where space was covered in different or similar ways drew his attention, firstly through symmetry and then with new forms, as he saw the consistency of a particular, asymmetric pattern on two sets of grids (the square grids and the $3 \times n$ ones). What makes the Husserlian approach for which I am advocating here for distinct from Cartesian and Kantian/constructivist views is that the object of the *"diagonals" configuration* that the intuition brought to the surface was *given* to him, in the overlap of his *lived*

[118] This tension between the body-subject and the body-as-object is also expressed in the well-known example of the hand that touches the other hand, which Merleau-Ponty (e.g. 2002, p.367) took over from Husserl ().

reality and his *embodied consciousness,* hence separated from his experience; rather than being established in his mind due to an internal representational capacity. Since "as soon as we distinguish this transcendental subjectivity from the living body (the "soul"), we get involved in something incomprehensibly mythical" (Husserl, 1970a, p. 118).

Immediacy—the non-mediated perception that brought the asymmetrical pattern to existence as a consistent form that may be applied to all grids, in the third step of the *intuition of essence*—is the key property that makes the intentionality fulfilled in this particular way, i.e. the intuitive way. It is what distinguishes intuition from imagination or remembering. And it marks the distinction mentioned earlier between Husserlian and Cartesian/Kantian approaches to the cognitive experience, since *immediacy* takes a new meaning under the Husserlian perspective: there is actual straightforward involvement in the world and *sensuous experience is already a form of understanding,* different from and indispensable for the positing, active constitution of the object as such. In other words for Husserl, understanding is not separated from our conscious—though pre-reflective—embodied processes, and the borders between Kantian *sensibility* and *understanding* are blurred and finally cancelled. In the detailed Husserlian/Merleau-Pontyian analysis of Ivan's intuition, immediacy was detected as operating through processes like the *stereoscopic vision* that I mentioned earlier (end of Part 3, §3.2.3), concerning Ivan's intuition of essence. What is particular about Husserl's theory is that immediacy in the sense just described brings with it an intentional act—a shifting of horizons that allowed Ivan to see the two examples, the square and the orthogonal grids as one, concerning the shape pattern, due to his intention for a unified pattern.

It is in the *third* step of the intuition of essence that the interweaving of the mental and the embodied is traced, not separated for Husserl, due to his *living body.* The *stereoscopic vision* brought the new object to the surface, an object of a new category, not involving any particular grids configurations and at the same moment involving precisely *any* grids' configuration. Since this is the other main finding of the analysis, namely the *generality* that the Husserlian *intuition of essence* invokes.

Ivan literally figured out a pattern that gave best results for any grid configurations and then he sliced the 2-dimensional space in sets of three, both in length and width, according to the *rhythm* of the arithmetic pattern (1, 1, 2, 1, 1, 2, 1, 1, 2,…). Ivan's transition from the arithmetic pattern to his formulas also has an embodied aspect that cuts across Kantian dichotomies, as well as current constructivist inspired research, based on Fischbein and adopting Piagetian ideas (e.g. the "cognitive conflict", Garcia, 1996, in Barach, Klein,), even when the attention is drawn on "the influence of the intricate interplay between intuition and logical thinking on the emergence of justifications and proofs" (Kidron & Dreyfus, 2014, p. 297). Since without the *intentional* organisation of Ivan's embodied conscious understandings (such as the allure that the rhythm of 1, 1, 2 evokes) into mathematical objects (such as the nine formulas), without the intentional history of the mathematical objects that Ivan devised being disclosed, the products of his cognitive process would become mythical constructions.

Ivan's intuition of essence was precisely the fulfillment of the sense of generality, found in the 'asymmetrical' pattern. Ivan managed to regulate the whole 2-dimensional space due to this intuition, and he was looking for nothing less. Intentional act and intended object are clearly manifested here, through the generalising positing of the pattern as the "pattern for all grids". The stereoscopic vision allowed the synthesis of coincidence. This abstract intuition is not only very effective but it also has an empirical dimension, as it can be taken up again and again, in different areas of mathematics, up to the point that evidence is brought to the surface, due to synthetic processes of confirmation and identification. But the principal perspective that makes it possible in the first place, as a meaningful experience of abstract, mathematically expressed objectifications is transcendental, rather than empirical; it is due to the interwoven empirical and transcendental ego, due to conscious acts of the intentional ego, in its diverse intentionalities that the abstract objects (finally) appeared. And the generalisation is a feature anticipated by Husserl's theory, considered as a particular feature of the intuition of essence. Immediacy was the manner through which an invariable feature was spotted (the shape pattern) and was *chosen* to express the configuration of all grids in the final

step of the process, where it was finally endowed with an intuitive sense of proof (*apodictic* quality), as the general pattern for all grids, after testing it in two instances (the square and the $3 \times n$ grids). And this sense of proof is the second feature of the intuition of essence, according to the Husserlian theory.

The living body was shown as preparing objects to emerge in the two stages of the intuition that preceded the final one of the *synthesis of coincidence.* The analysis has shown how the operative intentionality synthesised the pattern appearing in the $3 \times 3, 4 \times 4$ grids and persisted in it until it was posited as an object, namely the 'asymmetrical' diagonal pattern. And Diana transliterated her trial-and-improvement empirical method into a description of the rule of the task, in the shape of an algorithm. It is in this sense that the living body expands its territory of influence on abstract mathematical ideas, and supports new levels of abstraction and systematisation of primal material. , seen through the syntheses that it itself makes possible and the appearances of new objects, gives an alternative view of intuitions in relation to objectification.

3.2.3.2 The results of Ivan's successful intuition

The following drawings and e-mail that I received from Ivan give a description of what he did after he found the arithmetic pattern, as soon as he noticed the iterative structure of his shapes (Figure 6):

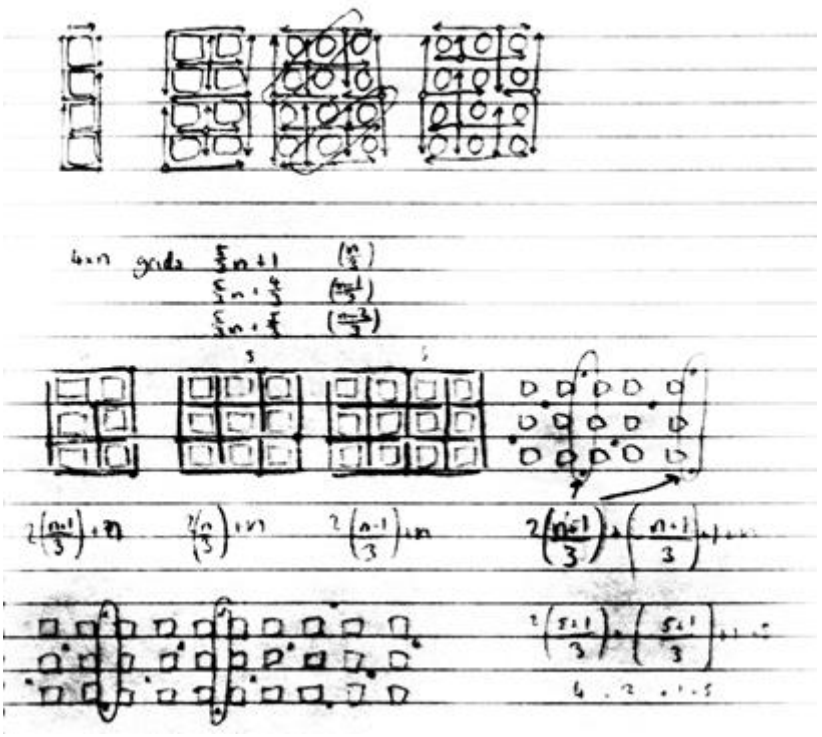


Figure 7. Ivan's building of formulas

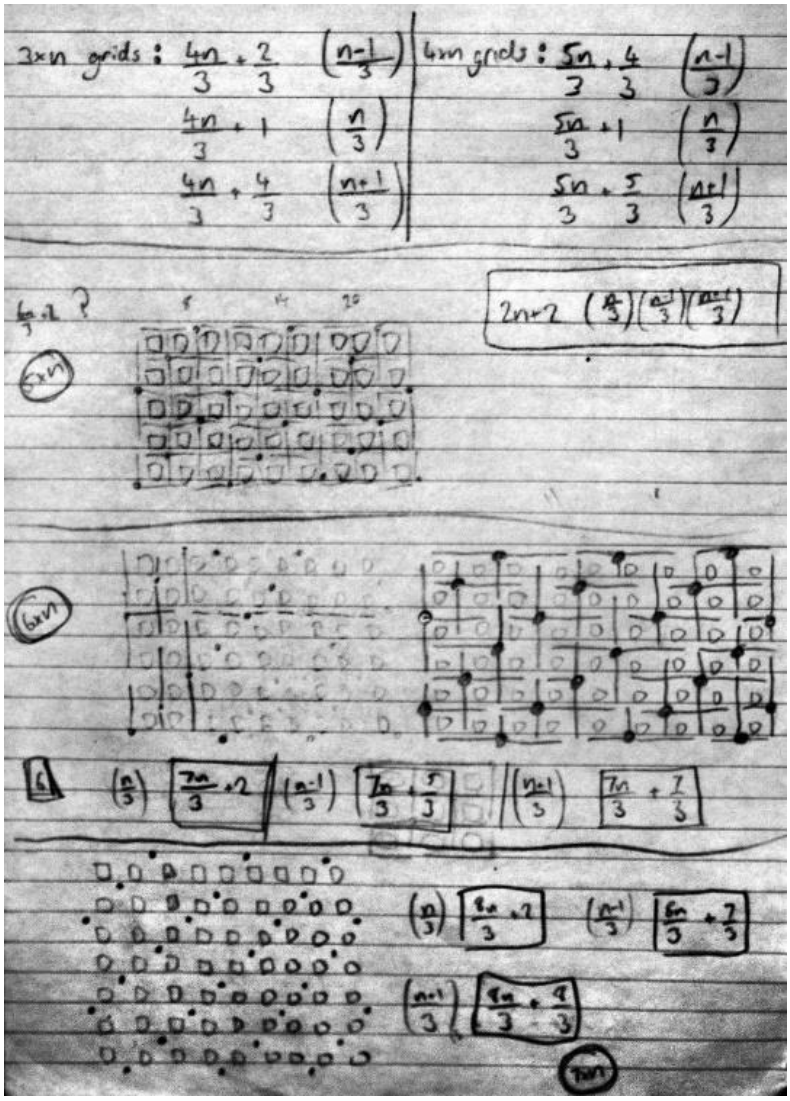


Figure 8. Ivan's building of formulas

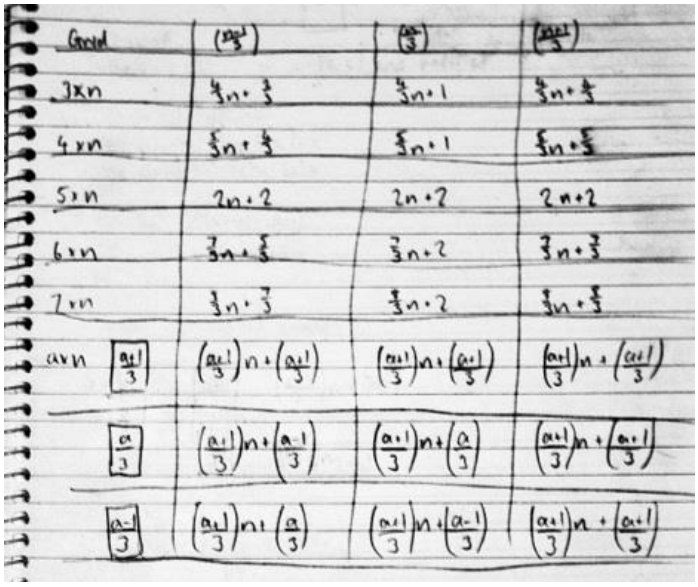


Figure 9. Ivan's final formulas

I realised that I could break it up into 3 sequences. Starting with a 3x2 grid, (3 up 2 along), [Figure 7] then the next grid in the sequence was 3x5, then 3x8... which gave me a pattern of 4,8,12,.... This pattern I called "when, (n+1)/3, is an integer".

I then applied this method to 3x3 (when, n/3, is an integer) to 3x6, 3x9... which gave me a pattern of 5,9,13,... and with the last sequence (when, (n-1)/3, is an integer) which starts on 3x4, then 3x7, 3x10... which gave me a pattern of 6,10,14...

This allowed me to apply so simple arithmetic sequence knowledge where the term no. was equal to the type of sequence that it was e.g. term 1 for the, (n+1)/3, set was worked out by using the formula, (n+1)/3, and substituting in 2 to get the number 1 (for term 1), when you substitute in the no. 5 you get 2 (for term 2), and so on.

I then applied the same thinking to the 4xn grids, 5xn grids, 6xn grids, and 7xn grids [Figure 8]. I then wrote out each of the formulas in a table, and this allowed me to see how the height can be related to the formula (which I called 'a'). This allowed me to generate 9 formulas based on when: (n+1)/3, is an integer; n/3, is an integer; (n-1)/3, is an integer; and when: (a+1)/3, is an integer; a/3, is an integer; and (a-1)/3, is an integer [Figure 9].

Chapter 3. Diana –feelings and enculturation in the emergence of an empirical intuition and the abstraction of a flow chart.

3.3.1 The course and the Doubling modulo activity

The nature of the course allowed many initiatives to every student, concerning the directions they might take, encouraging group and inter-group collaboration. The teacher adopted a merely operational instructional disposition during the sessions, which enhanced the effect of the individual/group independent methodologies and practices. The students had the opportunity to choose 3 or 4 of their favourite activities in order to further develop and present them in a written form at the end of the course. The minimised instructional character and the open-endedness of the activities facilitated the students' intuitive approaches to be more explicit than in many 'typical' conventional or exploratory teaching frames.

The task that will be analysed was called 'doubling the modulo'. The teacher would think of a certain number (the 'fixed number') and he would write a smaller number on the board, double it, and write the new result on the board, next to the previous one, if the result was smaller than the initial number. He continued this process until the doubled number would exceed (or become equal to) the fixed number. In this case the teacher subtracted the fixed number from the doubling number and the doubling process continued with the residue, the 'modulo', until he would find zero or a number that he had already used (a loop). He then started the same process again with any unused number smaller than the fixed one, until all numbers smaller than the fixed number would be recycled. It was an open-ended task in terms of the expected results, while the teacher prompted the students to pursue the strongest generalisations possible.

The students knew that the teacher would not respond to questions concerning the treatment of the task, and it was not always a case of a more or less smooth self-organisation or mutual adaptation in meanings negotiation and construction between the students. The teacher started the task by writing the first two or three numbers and then he was waiting for the students' response, writing the next

number either when the students got it right or after they couldn't find it. He did not give any clue, expecting the students to understand and share their understandings only by the repetition of the process. Luck [pseudonym] conveys his impressions of the introduction of the task until the "pattern finally clicked" for him:

> My head was spinning with numbers, I just couldn't connect each one to the next. I was doubling numbers without reason and trying to subtract any number that was in any way connected. Factors, multiples, divisors, nothing worked. My head was beginning to hurt when *a number was added to the board that was not part of the list* and yet connected all of them.[119] Not one of the listed numbers was higher of this number. In fact not one of the listed numbers was even equal to this, stopping just short. *I'm not too sure how long I stared at the list and the mysterious circled number before the pattern finally clicked.* That was when I truly felt that I deserved to feel slightly triumphant.[120]

As soon as some of the students had an idea they checked it and started sharing it with the other students. They followed different strategies while working in groups of 3, 4 or 5 members of their choice, and their strategies were related to the

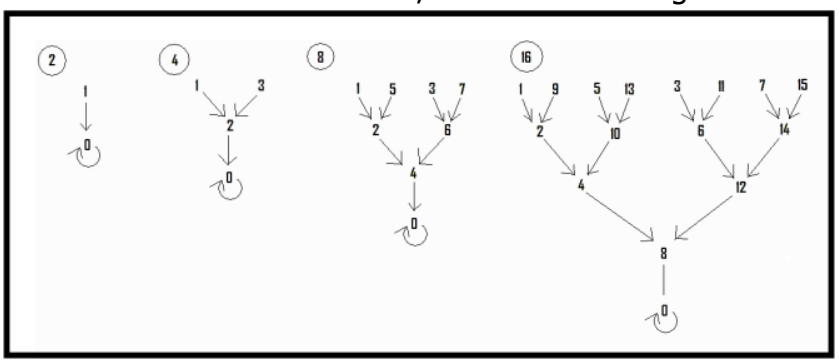


Figure 10. Luck's drawings for the activity.

exploration of the natural numbers' properties (i.e. even, odd, prime numbers,

[119] After the first examples when the 'fixed number' was not written or mentioned it was the first time that the teacher wrote on the board the 'fixed number' and put it in a circle.

[120] See Figure 10 for Luck's drawings for the activity.

powers of 2, prime factorisation etc.) and how these properties were related to the results (see Figures 10, 11).

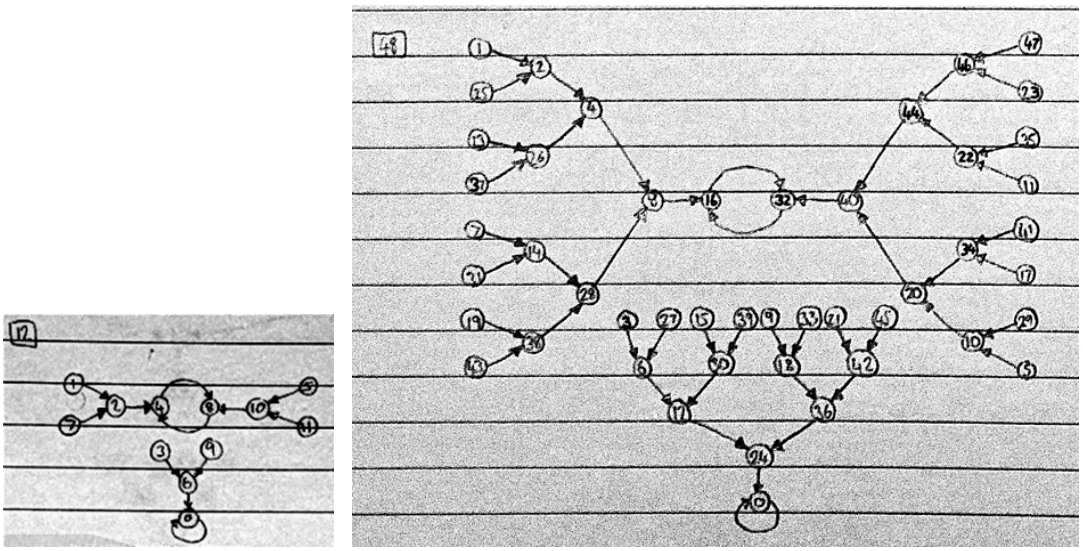


Figure 11. Ivan's drawings for $12 = 3 \cdot 2^2$, $48 = 3 \cdot 2^4$

Diana was trying to understand how the other students thought about the activity, without any success.

> I still don't understand how they do it; because a couple of people tried to explain it to me but I was just completely lost, I didn't know how they could look at a number and go "alright that the 1 so the next one needs to be 2 [or] okay that, the next one needs to be 4". I still don't understand it, their way. (interview extract)

The teacher did not provide any guidance, which is something that frustrated her:

> And sometimes it's just really *frustrating,* because I think, well, "am I doing it right or am I doing it wrong?" and he says "what do you think?" And I'm thinking that "I don't know, that's why I'm asking you!" And it frustrates me. (interview extract)

I noticed her frustration while working with another group, I approached her, I saw these 'strange' set of commands written in her papers (see Figure 12), and I realised that she was already engaged in a process of her own; thus I decided not to interrupt her and I chose to return to my previous group. The result was that she

developed her particular understanding alone. Her first intuition, as it came out in her coursework was to "look at the rule" of the task:

> I understood the rule and what we were doing but for some reason I couldn't get my head around the flow diagram that everyone was doing [Figures 10 &11]. I started looking at the rule we had been given and what we had found from the other numbers we had used. I decided I would use a flow chart and above it write out the numbers from 1 to n-1 [Figures 12 & 13]. This way I could see if there was more than 1 cycle and if there was more than 1 cycle I could cross the numbers out as I went along to make sure I had used them all.

The intuition of the 'task rule' has a particular significance, since it shaped her treatment of the investigation. The relevant data for this intuition are her notes and drawings during the session (e.g. Figures 12, 13), my observation during the session, her interview and her reflection after the session (delivered with her coursework at the end of the course). They deserve a closer examination, since they will be used for the phenomenological analysis that aims at the critical features of Diana's treatment of the investigation.

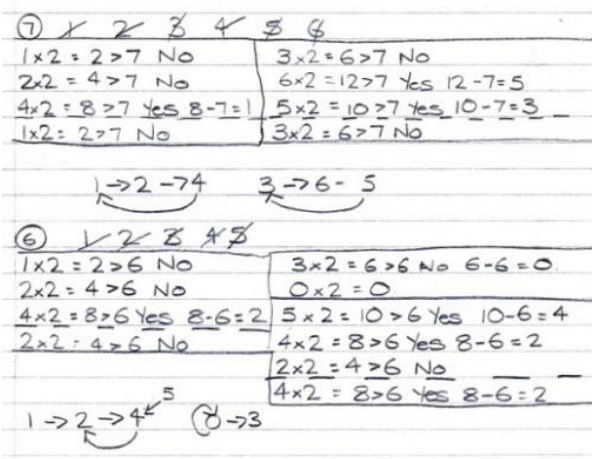


Figure 12. Diana's drawings during the session

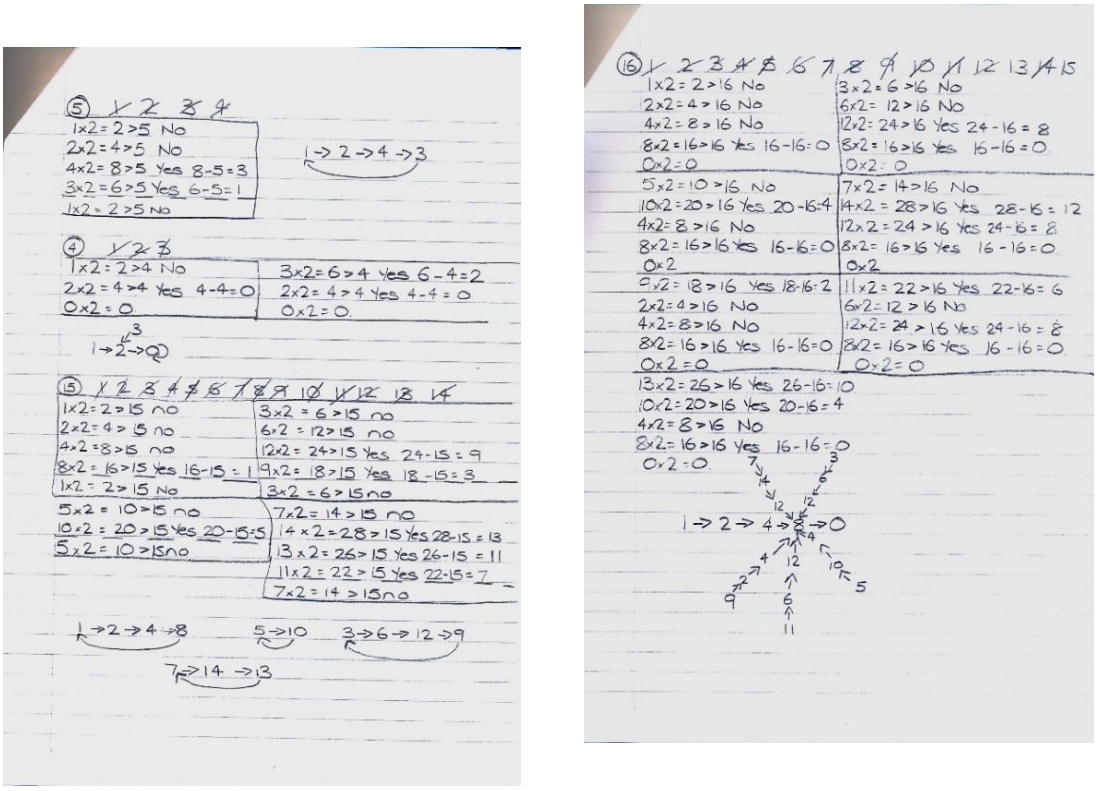


Figure 13. Diana's drawings during the session

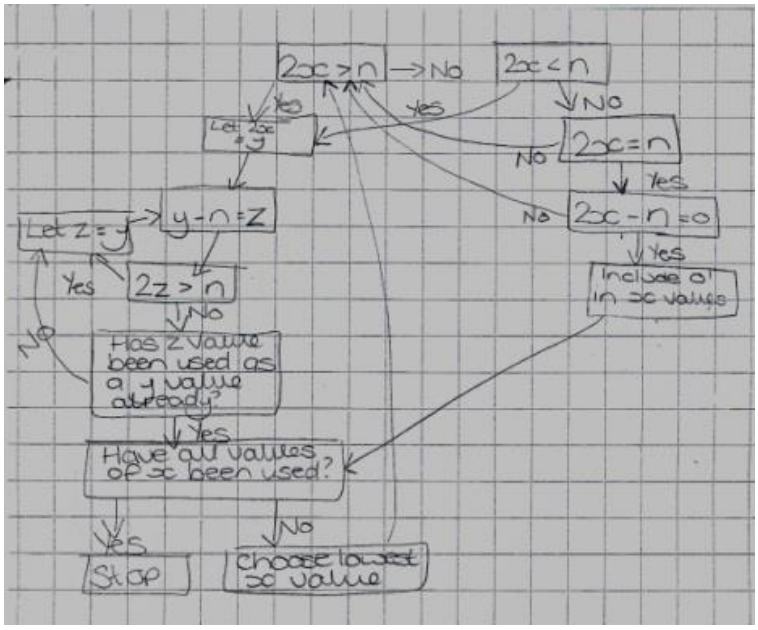


Figure 14. Diana's algorithm

She illustrated the structure of the activity ("the rule") in her own way (Figure 12), as a trial-and-improvement process, in a question/answer (q/a) manner:

> …so I was having to devise this kind of system, where I've said "right, okay, if I do **this** to that, if I times this number by 2, is it bigger than the 9? No it isn't! So what do I have to do to it? Then I have to do this to it". (extract from her interview)

We also notice her q/a approach in her text in Figure 12: Is 2>7? "No". Is 4>7? "No". Is 8>7? "Yes". Then, 8–7 = 1 (written on the same line as the q/a). Is 2>7? "No". A dotted line above the last q/a indicates that the latter part is unnecessary, since number 1 has already occurred. A 'loop' is completed and the corresponding numbers are crossed at the top. This series of q/a, written line after line for each number, gave full account of what happens when one doubles a number until one 'stumbles' on a number that has exceeded the fixed number or a number previously used (a loop), and her notes are a clear re-presentation of the actual raw judgments involved. The trial-and-improvement process that she used with *immediacy, intentionality* and a *feeling of certainty* is an *empirical* intuition according to Husserl, as it transformed her experience of her world-as-lived to successive questions and answers concerning numbers. Although the analysis could not go as far as finding an empirical equivalent of this intuition in her experience, we can only say that its crucial feature is the *separation* of the q/a sequence from her lived experience, similar to a situation such as 'finding objects in the dark'. It is an *empirical* intuition (since it draws forms directly from our lived reality) employing "[i]magination, for instance imaginary variation, [which] can serve the purpose of disentangling essences from the hyletic data"[121] (Hintikka, 2003, p. 177).

[121] 'Hyletic data' or 'hyle' is for Husserl the unstructured raw material that perception brings to our consciousness.

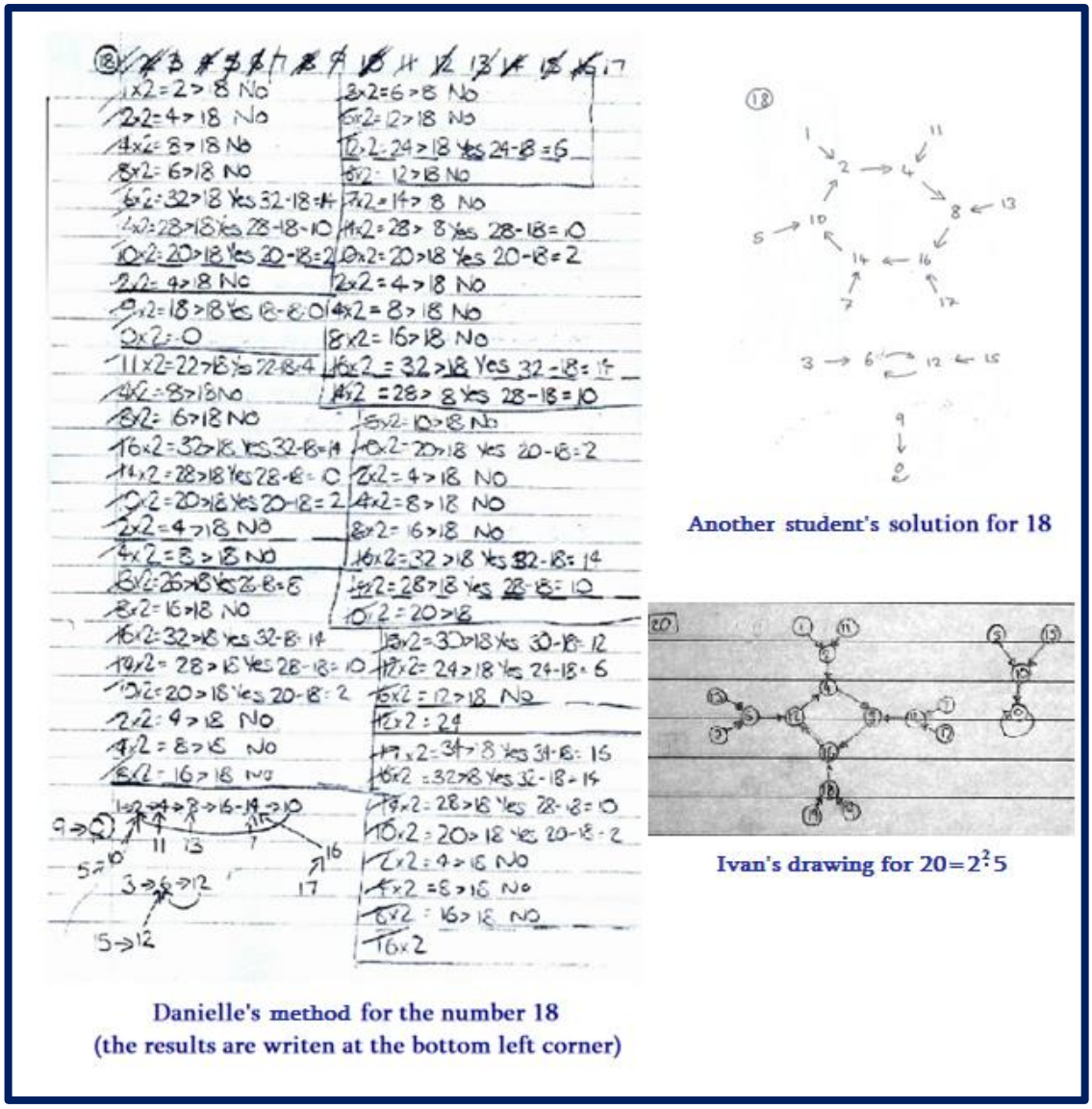


Figure 15. Diana's method, depicting what is implicit in other students' drawings

She filled pages of questions/answers sequences for each natural number from 4 to 18. Finally, these structured questions/answers became a new layer of data for her, and she intuited them as *commands* and *judgments,* similar to a computer algorithm. We see the *dialectics* of a new synthesis in the algorithm (Figure 14), which is the condensation of the 15 patiently repeated individual algorithms (drawings). These 15 drawings are prehensions (Tragesser, 1977, pp. 17-18) according to Husserlian theory adopted by this research study (Part 1, §§ 1.4.3.1, 1.4.5); The constitution of the algorithm that summed up the previous 15 drawings is a *new intuition,* which took place as she was attempting to describe the process she followed for the 15 numbers for *any* given number. It was an intuition since, as

Hintikka puts it, "according to Husserl such an abstraction of an essence can only take place in intuition", and in Husserl's words, "[s]eeing an essence is therefore intuition"(ibid, p. 181). This time her intuition was based on the previous empirical one. The q/a material was a result of her previous empirical intuition, being given in direct sensory awareness; but the intuition that followed separated the algorithm form from the q/a material, and it was what Husserl describes as the 2nd intuitive stage of the categorial intuition (the abstract stage), which in this case is also an intuition of essence *(Wesensschau):*

> [A]n analyst can separate in consciousness the forms *(eide)* from the material in which they are imbedded, turn them into objects of another sort, and … direct one's intuitive attention to them. This is what Husserl's *Wesensschau* is supposed to accomplish. This intuition has the effect of opening to my consciousness, not only an empirical object consisting of matter and form, but that form or essence in itself, separated from its *hyle.* (ibid, p. 180)

For Husserl the *categorial* intuitions—and the *intuitions of essences*—are the ones that actually create abstract objects, and Diana's example shows how abstract objects are formed on the basis of empirical ones, thus projecting an approach to knowledge that is essentially based on our lived experience, rather than internal mind processes. Diana's categorial intuition took place in two stages, since it started with the *empirical intuition*—the intuition of the 'task rule'—that separated the q/a form of commands from her trial-and-improvement lived experience (the first, empirical stage), and it was followed by her categorial intuition—the *intuition of the algorithm*—that separated the *form of the algorithm* from the 15 drawings (the new 'hyletic' (raw) products) of her empirical intuition (the second, abstract stage). The new object, namely the algorithm was an object of a new class (a new category), leading her previous results to mathematical abstraction and formal language. But as I just mentioned there is also another way to interpret her new object, i.e. as the *essence* of her 15 drawings, which transformed the 15 "sensible particulars" to something different and essentially abstract. Hintikka's explication of Husserl's *intuitions of essences* finds here a clear instantiation since, "*Wesensschau,* too,

pertains to objects, albeit objects different from sensible particulars. For Husserl *Wesensschau* transforms sensory intuition into intuition of essences" (p. 182).

3.3.2 The flowchart and the legitimisation of her efforts by the teacher

In understanding Diana's actions and the origins and evolution of her material the phenomenological analysis did not raise questions concerning the effectiveness of the produced objects for the task at hand, from any other perspective apart from the student's perspective, as she was intentionally driven towards the object (the algorithm), recognising at it an effective artefact for the task, as it was perceived by her. We followed the object's genetic course, as it emerged from the learner's constitutive acts, namely from her intuition of the 'task rule', her ever closer apprehension of the rule of the task in her 15 drawings that followed (considered as prehensions that led to the next intuitive apprehension) and finally, from the *intuition of the algorithm,* which brought the algorithm to the surface of her conscious investigation, thus concluding her efforts during the session.

The algorithm was the state-of-the-art product of the student's investigation during the session, but Diana was not too happy about her results, since she was convinced that no one would be able to understand them, let alone the teacher, who would evaluate the productions that the students would choose as their favourites at the end of the sessions,[122] as this one certainly was for Diana:

> Well, when I was looking at this, I was thinking if I was to give this to the teacher, he'd look at it and he'd be thinking "what on earth has she done?" Because he would not understand it! So I thought the only way I can do it is by doing some kind of flowchart, flow diagram or algorithm or whatever to be able *to explain my steps in the thought process that I went through, to be able to understand this activity*… (interview extract, her emphasis)

Since the bracketing of the teacher in my research goes as far as the teacher's actions did not interfere with the student's acts and the objects that were

[122] Up to 8 different reflections on different tasks were delivered by each student at the end of the course, which was about the time that my interview with Diana took place.

constituted due to these acts, there is an occasion here that the teacher—as any other teacher, and despite this teacher's masterfully performed withdrawal from instruction—is vital, since he is responsible for the legitimisation or the rejection of the students' constitutions and constructions. The teacher's legitimisation of her (coherent) mathematical description, was critical for the impact that the task had on her. The appreciation of her way of managing the activity was also a motivation for her, as she explains in her interview, saying that she made the flow chart in order to be better understood by the teacher, as she was aiming to deliver the description of this activity as one of her final assignments (her favourite one – see interview).

But the other side of the story is the transformation that she went through, in order to alone construct an algorithm, after her 15 empirical attempts. It was by managing "to understand this activity" that she arrived to the algorithm (Figure 14); the algorithm was explicative of her "steps in the thought process". The algorithm was depicting the steps that she "went through, to be able to understand this activity".

What we are basically told by the student is that she transformed her trial-and-improvement method to sets of questions/answers, due to her empirical 'intuition of the rule'; and that she then intuited her 15 applications as a generalising algorithm, at the second stage of her *categorial* intuition. And even in her final object (the flowchart, Figure 16) the "thought process" is visible, hence the object becomes a re-presentation of the initial trial-and-improvement method.

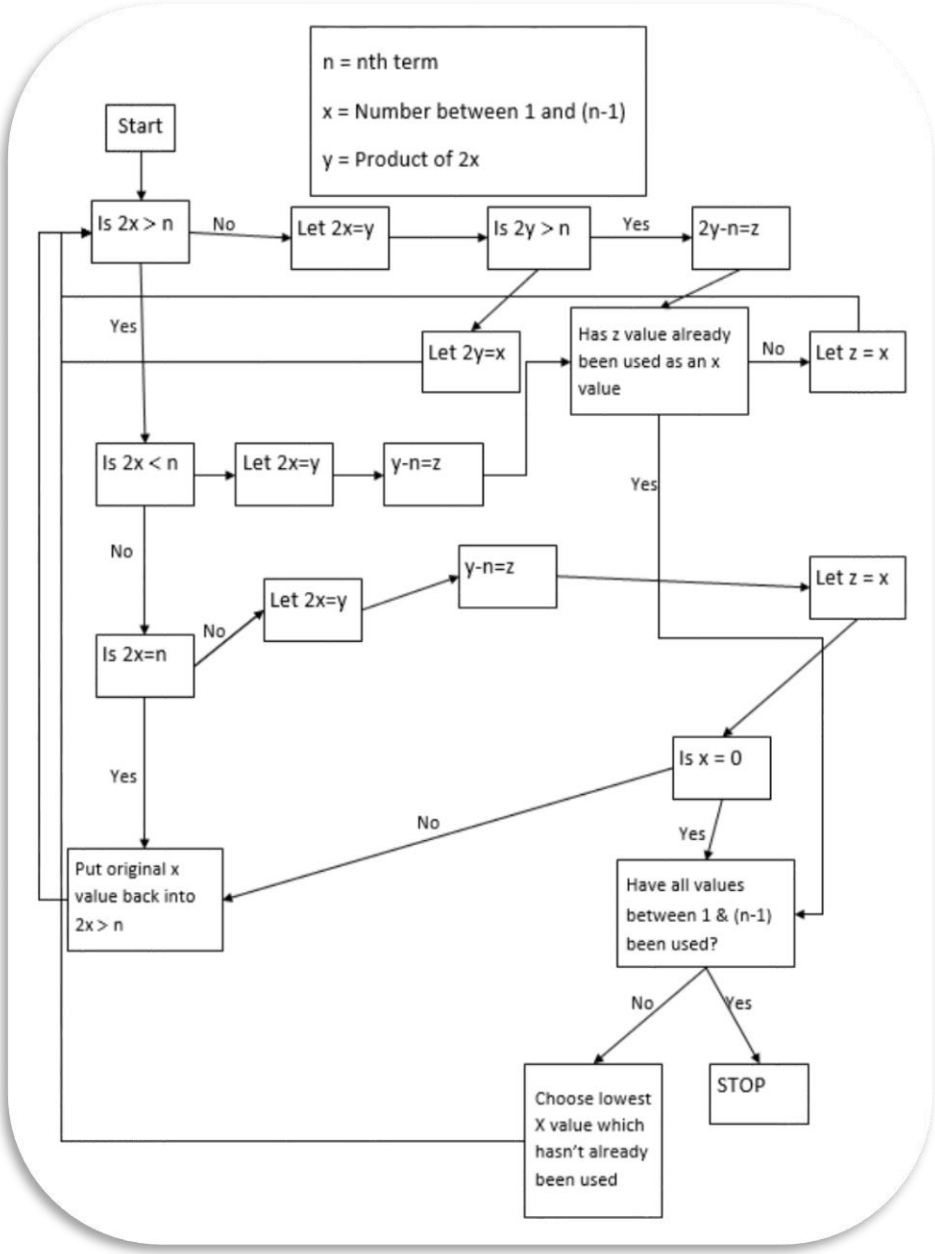


Figure 16. Diana's flowchart

Exploring the *intentional origins* of the transformation of the *algorithm* (that she created at the end of the session—Figure 14) to the flow chart (that she designed when she went home—Figure 16), also serves as a testification of her originally *intersubjective* intentionality. The flowchart indicates and moreover designates intentional horizons that remained invariant throughout her investigation, which can be brought to light by the phenomenological analysis, although they are not my primary focus of investigation: under these horizons the

flowchart can be seen as an *extension* of her intention to constitute something intelligible for others as soon as she 'decoded' the rule of the task, during the session, and alongside depicting and thematising the 'rule' of the task fifteen times on paper. Her intention to convey her ideas to others (and in particular to the teacher), led her to the conversion of her work to more intelligible forms (according to her). Thus she expressed her need to better present her investigation findings, which was an ever appearing process, from the beginning until the end. Her interview,[123] performed in the open and spontaneous frame that Diana set (and which I also encouraged), with her marvelous sense of communicating her ideas is telling, up to its details…

> R: The flowchart is much more advanced than this one [the algorithm], although it might describe the same thing, but the flowchart is far more advanced, I think.
>
> D: Well, *I knew* that if anyone else other than me looked at this they wouldn't know what was going on.
>
> R: *I* did!
>
> D: Oh, did you? Well I just *assumed* then, shall I say, that if *anybody looked at this* they wouldn't [know] what was happening and how I'd got my answers; which is why I did that [the flowchart][124]
>
> R: That is far more sophisticated!
>
> D: It's a little bit confusing though, isn't it?
>
> R: Why?
>
> D: It's easy to follow but there's just lines everywhere [laugh]
>
> R: [laugh] is it your thought depicted here?
>
> D: Yes!
>
> R: Is this precisely how you have to think and how you thought…
>
> D: *That's how I thought* to be able to do it.

---

[123] The whole interview can be accessed in the Appendix.

[124] Diana is emphatically repeating that her *motivation* for advancing her initial 'sequence of commands' to a flowchart was *the need to be understood by the other people.*

R: It's a bit of a maze, isn't it?

D: It is [laugh] but...

R: [But] there is an outcome, somewhere, isn't there?

D: *To me that makes perfect sense!*

R: Yes, yes, absolutely, I like this one… I'll photocopy it actually[125]

D: Yeah!

R: I like it very much.

D: Well, I've got on my… I can e-mail it to you, if you like.

R: Oh please, yes.

D: Because I've got it on my computer; so if it's easier I can email it to you.

A: Oh yes, that would be even better.

D: I still need to double check it because again it's a while ago since I've looked at this;[126] and I did double, triple and quadruple checked that this worked and it did, but I'll double check it again [laugh], just for my own benefit.

R: Alright, I would accept it as it is, even if it had errors,

D: Yeah!

A: I mean it doesn't make a big difference to me; to me it is the construction that matters.

D: Right, okay.

R: And I can understand what you're trying to do here; so, that was about it with this activity, which I call it 'rings'… the doubling

D: I think it was called 'doubling', yeah…

R: Doubling, yes, 'doubling modulo'.

---

[125] My statement is formed as a question—asking for her permission to photocopy the flowchart.

[126] The interview took place at the end of the sessions (11-4-2011), five months after the *Doubling modulo* session (11-11-2010). But her recollections are expressed without any sense of doubt.

D: Yeah but the reason… *I did quite enjoy that one but that was because I had a different way to everybody else; that's why I liked it* [laugh][127] (12:20)

R: Do you usually have a different way to everybody else or do you usually have the same way with somebody else?

D: No, usually my thought is… my thought is the same as everybody else's, so *this one quite intrigued me* because I thought it in a different way than nobody else seemed to, so it intrigued me, because of that.

It now makes better sense what Diana expressed in the previous extract, concerning her "steps in the thought process that I [she] went through", when she emphatically states in this extract, that the flowchart illustrates *how* she *"thought to be able to do it",* when I asked her if her thought is depicted in the flowchart. My bracketing of the teacher due to my focusing on the primal constitution of each key object in the students' learning experiences does not turn me blind to the social aspect that this particular teaching made available, especially when the student herself raises the issue of the uniqueness of her treatment of the *Doubling (the) modulo* task . This uniqueness also manifests the uniqueness of *a teaching approach that is tolerant to different approaches by the students*; the teacher's withdrawal from instruction, and his broad acceptance of the students' mathematical suggestions, matching with a good sample of students and a joyful disposition towards the mathematical investigations of the course, with me as an unexpected attraction of the sessions, are precisely the main reasons why *these courses, organised by that teacher, that particular academic year, shaped a particularly hospitable milieu for my theoretical and methodological research approach*; and allowed my phenomenological methodological instruments to be implemented.

It is in this sense that the teacher's legitimisation of Diana's object—the flowchart—was pivotal for Diana's success. It is clear that Diana's "*different way to everybody else*" was exactly what she needed, and she acquired it not only due to her efforts but also due to the teacher's open attitude towards the students'

[127] The short laugh comes after a statement that was uttered very seriously. All italicised words are due to her emphasis, apart from the one related to this footnote.

mathematical suggestions. It was exactly what she needed in order to build up a more coherent and self-confident mathematical interface.

> D: Yeah but the reason… *I did quite enjoy that one but that was because I had a different way to everybody else; that's why I liked it* [laugh][128] (12:20)
>
> R: Do you usually have a different way to everybody else or do you usually have the same way with somebody else?
>
> D: No, usually my thought is… my thought is the same as everybody else's, so *this one quite intrigued me* because I thought it in a different way than nobody else seemed to, so it intrigued me, because of that.

I can easily think of many classroom contexts—including Diana's own school experience (as it comes out from her interview - see Appendix C)—where Diana's devise would be deemed inadequate or irrelevant.

3.3.3 Diana's presentation at the end of the course.

3.3.3.1 Diana's presentation – A bold decision to go against her previous conceptions

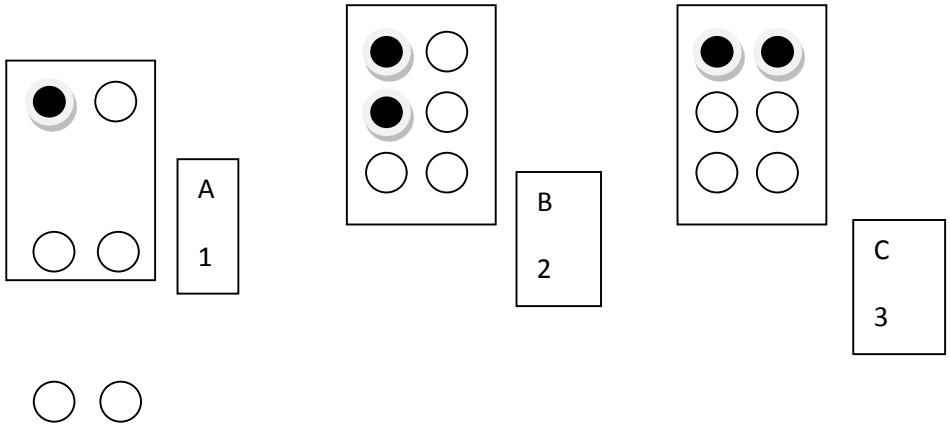
The last three sessions of the NoM course were dedicated to the students' presentations of a task of their own choice, while the other students, the teacher and I participated in the investigation. The intention for these sessions was to give the opportunity to reflect on what the students possibly gained from the sessions, and express it from the 'other side', the side of the teacher. Each student's presentation lasted 20 minutes, followed by 10 minute discussion concerning the presented activity. Again, no rules were prescribed on what they might choose to present, no intervention by the teacher or me, no ideas introduced, and the students were left to choose whatever they wanted, in order to present it in any way they had chosen.

The teacher organised a session before the aforementioned students' presentations, with a lot of books that could offer many ideas for mathematical

[128] The short laugh comes after a statement that was uttered very seriously.

activities. The ideas had a suggestive rather than an obligatory character—Ivan for instance chose a theme that was inspired by one of the books that he had at home. On the contrary, Diana chose to present a task from one of the teacher's books, related to the Braille system. For the description of the task I'll use Diana's own words, from the coursework that she delivered at the end of the course:

> Louis Braille completed the Braille system at the tender age of 15. It was introduced as an aid for blind people to be able to read. The system is composed of a 2x3 rectangle with 6 dots. When the dots are raised it represents a letter, a word or sometimes a number. The diagram below shows a couple of examples; the black dots represent those which are raised.



> The idea behind this investigation was to see how many different possibilities there were if there was 1 [black] dot in the grid then 2 dots and so on and so forth. I decided against looking into this myself so I could learn with the class when they were doing it. Another reason for not knowing what would happen was so that I had no preconceived ideas when it came to how people worked out the answers.

Diana sets the target of the task when she writes that "The idea behind this investigation was to see how many different possibilities there were if there was 1 [black] dot in the grid then 2 dots and so on and so forth".

She explicitly mentions that she "decided against looking into this myself", and she justifies it in two ways:

- That she wanted to "learn with the class when they were doing it"

- That she would have "no preconceived ideas when it came to how people worked out the answers", so that she would not intervene during the process, by giving straightaway answers.[129]

The black dots signify raised dots, i.e. tiny palpable bumps, in order to be identifiable by people with vision difficulties. And the task that was introduced by Diana to her peers, the teacher and me was to find out how many different positions of black dots are available for the Braille system, that is how many different positions of 1, 2, …, 6 black dots exist. The mathematical ready-made answer concerns the computation of the permutations expressed by the formulas $\frac{\mathrm{n!}}{\mathrm{k!\cdot(n-k)!}}$, where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, n equals to 6 and $\mathrm{k} = 0, 1, 2, \ldots, 6$. But the mathematical ready-made answer is not in focus here. What is significant from the point of view of the study is that Diana did not fix the results of the task in advance nor did she have "preconceived ideas" concerning how people should work out the answers. She decided to go *"against looking into this"* herself and she used the freedom that the course offered, by following one of the course's tenets, namely that the teacher "learn[s] with the class", rather than imposing the teacher's mathematical model to the students. The second reason she mentions is related to the first, since by "not knowing what would happen" she would have "no preconceived ideas when it came to how people worked out the answers"; therefore she would refrain from giving the answers "straight away". It was obviously one of the things that had stood out for her, especially after her 'doubling modulo' experience, where she came out with her unique answer, although no guidance was provided by the teacher nor the help she received from her peers was 'adequate', in order to advance by using one of the methods that they did.

> D: I think when I'm teaching I need to get over the urge to give answers… not straight away. If somebody doesn't know the answer to something I'm very… I *want* to be able to tell them rather than help them to understand.

[129] See the next section for an exploration of this feature of her teaching attitude, and its subversion due to the sessions of this course.

R: [You mean] to understand themselves?

D: Yeah, I sometimes have an urge to say "the answer is this"; rather than try and explain it. (interview extract, her emphasis)

And she consciously decided to use the NoM strategies against her previous conceptions of teaching and learning.

3.3.3.2 The answer-feeding teaching method and the impact of the NoM sessions

Diana articulated in her interview the mixed feelings she had before, during and after her presentation. One of the issues that were critical in understanding her presentation is related to her tendency to provide the learners with answers. Let us see how Diana described this issue, her frustration when she did not have any guidance during the NoM sessions, and the influence that she acknowledged to have received from the NoM sessions and her own presentation:

D: I think when I'm teaching I need to get over the urge to give answers… not straight away. If somebody doesn't know the answer to something I'm very… I *want* to be able to tell them rather than help them to understand.

R: To understand themselves?

D: Yeah, I sometimes have an urge to say "the answer is this"; rather than try and explain it. I am controlling the urge I am doing quite well with that; I'm having training when I'm teaching [laugh] but I… but I don't know... I don't really know because I have to admit in the NoM sessions I have sometimes been *really frustrated* because I sometimes felt as though… *I've asked a question and he's* [the teacher] *answered it with a question!* And sometimes it's just really *frustrating* because I think, well, "am I doing it right or am I doing it wrong?" and he says "what do you think?" And I'm thinking that "I don't know, that's why I'm asking you!" And it frustrates me

R: [laugh] Yeah, all he means I suppose is "do whatever you want…

D: Yeah, yeah

R: …it's all about your way of doing it, not me telling you how to do it"

D: Yeah, I understand that, *I do understand that but I had times that I found it really frustrating.*

R: Because you need somebody to really tell you *the answer?*

D: Yeah, because I feel I've got to a point where I *really* don't understand it anymore and I don't necessarily need an answer! Just a little bit of *guidance*; maybe *guidance* would be a better word rather than answer, you know, *a push in the right direction;* but sometimes I felt as though I wasn't getting that and it frustrated me.

… … …

So it has, yeah it's been good for me! *I have enjoyed it but it's been a hard work* [laugh]

R: [laugh]

D: It's taken me a while to fully appreciate it! Let's put it that way.

R: Do you think that you will apply any of these things to your teaching?

D: Yeah, well, like I said before about the… about my need to just give people the answers but… I think *I have realised from these lessons that it helps people more if you help with their understanding of it rather than just give them an answer straight away;* and naturally *make them think about it and what they are doing and why they are doing it,* which obviously is very important with wanting to be a teacher. It has been good! Yeah!

R: After so much 'pain and suffering'…

D: After so much 'pain', yes! It was worth it[130]

In order to understand what it was that changed Diana's mind about her "*need to just give people the answers*" we will start by exploring her mixed feelings about

[130] See the text around footnote 89 in 'Diana's transcribed interview 150411'; by saying that it was 'pain and suffering' I actually repeat what she said a while ago.

the NoM sessions.

Diana expresses her mixed feelings for the task that she presented:

> Well, partly, the activity was alright, I was doing it and it seemed like a good idea; I felt okay about it before I started doing it and then I started doing it and I just thought "this is rubbish"

And she explains how she resolved the 'problem' of not being able to provide guidance to the students' findings…

> because I hadn't put as much thought into it [the Braille system task –her presentation] as I should have done, which I'll admit, I completely admit I should have put a lot more thought into it than I did; and I never thought to see the pattern,[131]... and then obviously I saw a pattern in it but… in a way I was just going with the flow of the class and what they found because I didn't really have any answers to it,[132] *I didn't know what was going to happen!*[133]

…and how she arrived to a new position, different to the answer feeding model…

> In a way I liked it that I didn't know the answers because I was learning with the class. And it was nice because I had certain ideas of… sort of a rough idea of what might happen but then it was interesting to see everybody else's ideas; so in a way I suppose it was a good thing because I had no preconceived ideas of what might happen, it left me open to all the possibilities, so I wasn't saying "right, well this is the answer to it"; it wasn't this definite answer, it was like "right, like okay let's just explore it together

---

[131] She means that the results for the task that she set followed a pattern that the students discovered during her presentation. The pattern is also revealed from the formula $\frac{n!}{k!\cdot(n-k)!}$, since the results are symmetrical (palindromic): (k,p)={(0,1),(1,6),(2,15),(3,20),(4,15) ,(5,6) ,(6,1)}, where k is the number of the 'raised' black dots, and p is the number of positions of these k tots for the total number of 6 dots; The just mentioned solution is a more general mathematical articulation of the answers, since the case (0,1) does not appear, since there is always at least one raised dot.

[132] See means the students' findings; she did not have ready answers or guidance for their findings.

[133] For the previous extract see the footnote 61, for this extract see the text before the footnote 60, and for the next one see the text before the footnote 63 in 'Diana's transcribed interview 150411'.

and see what happens". So it left me open to suggestions.

…since she allowed herself to come to the position where she could not say "right, well this is the answer to it".

It happened in the same way that she was 'forced' to devise and construct her algorithm, because "everybody else seemed to be able to say the next number but I couldn't, so I was having to device this kind of system". This time, she was 'forced' to acquire a new strategy towards teaching, since she had to avoid providing answers, in order to be in conformity with the NoM setting—as she thought of it—where the presentation was supposed to take place. She 'burned the bridges down' by not "put[ting] as much thought into it", by not "know[ing] what was going to happen". She forced herself to think with the others until she saw the pattern that they uncovered. As Badiou puts it:

> thinking is never a matter of voluntary decision or natural inclination. We are always, he declared, *forced* to think. Thought pushes us, as it were, from behind. It is neither lovable nor desired. Thought is violence, done to us. (Badiou, 2011, p. 110, emphasis in the original)

She was forced to adopt the strategy that she followed during her own presentation: "I was just going with the flow of the class and what they found because I didn't really have any answers to it, I didn't know what was going to happen!" And this strategy was to "explore it together" with the class "and see what happens". Shifting from the student to the teacher role gave another perspective to the previously felt frustration. But this realisation did not come alone; it followed the realisation that her constructions are just different than those of other people rather than wrong:

> R: What have you discovered from these sessions? Have you discovered anything in your personal way of…
>
> D: I think, *personally*, when I put my mind into it I can actually come up with things myself rather than relying on other people and saying "well, what have you got for this and what have you got for this"; and I think I've learned to trust myself a little bit more. Because a lot of the time –and it's not just with

maths, it's with other subjects- I've been very cautious when I found an answer, if my answer isn't the same as everybody else's I've just assumed mine is wrong! Because it's not the same as somebody else's [answer].[134] But I think the lessons that we've had and because of the way that we have been taught to think for ourselves, and it is a lot of independent thinking, that I think I've possibly become a bit more confident with what I'm doing; because I'm thinking "well, because I've got a different answer to somebody else it doesn't necessarily mean that it's wrong, it just means I thought about it in a different way". So it has, yeah it's been good for me! I have enjoyed it but it's been a hard work [laugh][135]

Her realisation originated from another survival moment, i.e. from her devise of a very personal method of managing the 'doubling modulo' activity and the construction of mathematical objects, namely the algorithm and the flowchart.

So how one survives when she thinks that she should provide answers and she does not happen to have any? When the students have the initiative and they surprise the teacher by producing results that neither the student nor the teacher had expected? The context of the Nature of Mathematics (NoM) course was not in favour of supplying guidance, as Diana had experienced in an unpleasant way, but she adopted a NoM approach during her presentation. It was probably not something that she had predicted but it seems like it was something she was ready for. And, as she reflects on her previous frustration she relates it to the more or less traditional way that she was used to:

R: And I was just asking how you feel how you think about this alternative way if you like,[136] now that the sessions are over; how you feel about this alternative way of learning and constructing knowledge

D: I think it's good because *encourages you to learn for yourself* and *you can then see how you learn* and apply it to other things; other than just maths

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[134] My emphasis.

[135] See the text around footnote 91 in 'Diana's transcribed interview 150411'.

[136] That is the NoM way of learning and constructing knowledge.

you can apply it to different things whereas… I'm thinking *in schools and everything it is as we've spoken about before you are not developing necessarily your understanding, you are been taught to pass an exam!*[137] So, I think it was… *I think part of it I found it frustrating because I never had this kind of teaching before; throughout college and throughout school it's alright "you need to do this to be able to get this grade, you need to be able to do this to be able to get this grade" whereas this was a lot more of my own thought processes and understanding and…*[138] So it's good but I think because it's new and maybe because I'm a bit older and I've never come across it before

R: A bit older? So you think the others have come across it before?

D: Not necessarily! I don't think Donald or Ivan have ever come across it before, but because I've got through say 9 years more teaching than Donald and Ivan for example, then, you know on different teaching methods I've still never come across it. So, maybe it's harder for me than it is for them because they're more… open to it?[139]

But what I thought, especially after the interview was that Diana was indeed open to it!

Chapter 4. Mary – The representative case of all detected intuitions and objectifications

An introductory example

My observations aimed at 'moments' when objects were brought anew in front of the students' consciousness; I aimed at such objects as they appeared for the first time, attempting to track such 'moments' in any way possible, and as close as possible to the time that they took place. But my presence was not facilitating such moments, since I often became a point of reference, mostly due to the students'

[137] My emphasis.
[138] My emphasis.
[139] See the text before footnote 89 in 'Diana's transcribed interview 150411'.

attempts to extract information concerning the tasks. The tracking of a moment when an object appeared for the student as new gave me a rear feeling of excitement and achievement, as if I was next to a realm of untold richness and mystery. Such was the feeling I had when I read Mary's reflection on one of the activities early in the course, when I was not yet familiarised with the students. The activity was about building solids, such as the stellated octahedron, and I'm using it here as a first taste of the data that will be analysed in this chapter:

> I knew in this shape it was going to be made of triangles, and was only going to have a base, with no top if you like, or a top with no base. So we set out cutting out the triangles and squares to make the square based pyramids so it was going to make it easier to visualise. It took us a while but we got therein the end and we began to stick the shapes together. During this task when I first suggested it was told "that's going to take a really long time". My response was "We have all the time in the world Yvette!" I genuinely thought we were going to need it to accomplish this.
>
> From working out the octahedron found that creating a net first made it so much easier to stick together at the end. This is what me and Barbara chose to do where the other groups just started sticking. In my head I had a picture a star with a square on one side, it wasn't a shape it was just another piece of crazy imagination. And by the time we had finished our net it was so far from the shape in my head it was laughable.
>
> We'd gotten to a net that looked like some sort of weapon from Robin Hood, and we were just getting ready to stick it together to make a solid when BAM time was up. The other two groups had finished and the teacher performed his magic of placing the two squares of the two shapes the groups had created and before your eyes the stellated octahedron was born. He asked "does anyone know what this shape is called?" The general consensus was a star, he corrected us with the correct term. It's not something I'm likely to forget again. And in a way it was lucky [that] Barbara and I never finished building our shape as we then had three stages of creating the "star"; we had the net, the solid that would fill the hole in the large square based

pyramid, and the final "star". It was indeed a revelation.

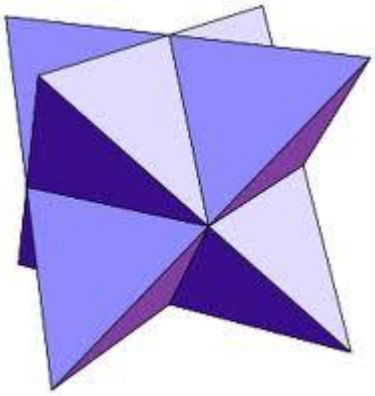

Figure 17. Stellated octahedron. Each group constructed half of this shape.

A sense of immediacy is felt in Mary's reflection (written the same day that the activity took place), one that only a re-lived experience can manage to bring to the surface. Such moments, even though I hadn't managed to record them at the moment that they took place, brought due to their vivid reflection clear evidence of the constitution of objects by the students.

3.4.1 The description of the course – The case of Mary

The data analysis that follows is a *case study* from a course that took place in the academic year 2010-2011 and involved a group of 13 students, training to be teachers in British secondary schools. The course consisted of twenty 3-hour sessions, where the main targets were the production of generalisations by the students and their diverse conceptualisations of mathematical phenomena, such as the embodied understanding of curves. The teacher avoided the provision of information concerning the tasks, intervening in the students' methods or interrupting their discourse (as described by Brown, 2012); he deliberately and explicitly limited his interaction to the description of the operational aspect of the tasks. This was one reason that this course was particularly appropriate for my research, since I intended to look at the learning process from the first person perspective and to bracket (i.e. set aside from the present consideration)

influences of cultural tools and practices, which inevitably accompany any education. [140]

The students were invited to use their common sense and mathematise their results in any way they thought appropriate. They were working in small groups of 3 to 5 people of their own choice and it was suggested to continue their investigations at home and push them as far as they could in order to submit their findings in their coursework.

In the episode that will be analysed[141] the students were asked to make sense of a curve through exercises centred on the students' bodily movement. The teacher of the course asked the students "to think how they can move whilst remaining equidistant" from the wall (10 foot-paces) and a student who remained stationary. No other direction or indication on how to proceed was given to the students, who were free to handle the activities any way they thought suitable.

The student of this *case study,* Mary (pseudonym), was embodying the fixed point, while another student was acting out the curve (Figure 18). I participated in Mary's group while avoiding to facilitate the students' investigations. A few days after the session Mary presented an elaborate drawing (Figure 19) and a detailed account of her treatment of the activity at home. At the end of the course she submitted her report on this activity as her favourite amongst several.

The session was audio and video taped (after gaining the students' permission) and Mary's home work was analysed through interviews and the

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[140] It doesn't mean that I suppose intuitive processes to be limited to individual processes: I acknowledge that for the fullest understanding of intuitive processes in the learning practice attention needs also to be paid to the social-cultural dimensions, as it is attempted in Andrà & Santi, 2013, Andrà & Liljedahl, 2014, although these dimensions are bracketed for the purposes of this study.

[141] A different perspective of this session appears at a paper that was published recently (Brown, Heywood, Solomon, & Zagorianakos, 2013).

text of her coursework. An interview was conducted a few days before the student delivered her coursework, then an additional interview took place and an e-mail was used in order to clarify a few issues. In the following sections I analyse Mary's data, as they were triangulated or cross-checked from the aforementioned data sources.

Figure 18: The other student with her back to the wall and looking at Mary (the fixed point) moves her hands perpendicularly towards the wall.

3.4.2 The three stages of the student's investigation

In further analysing the empirical data my phenomenological method allowed essential structures to be revealed by acknowledging the learner's intentional transcendental ego as an objectifying unit. In the course of close exploration of all the material three stages emerged from Mary's investigation, namely the *embodied* (when she was in the classroom) the *diagrammatic* and the *formal.*

3.4.2.1The embodied first stage of Mary's investigation

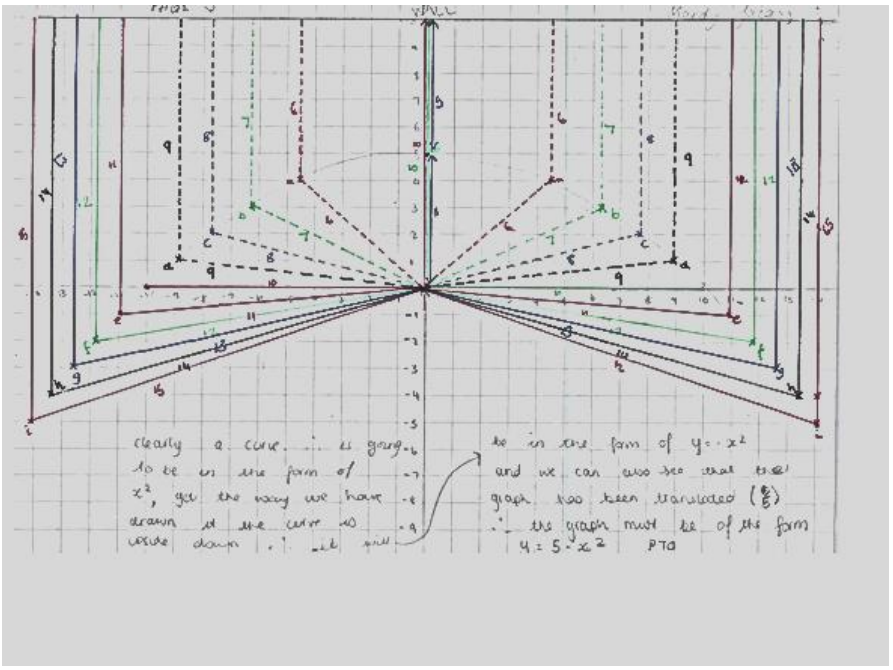
The analysis of the audio and video recordings of the first stage revealed important clues about the curve and her formal negotiation that followed:

- Mary, who acted as the fixed point, thought that the curve was a straight line parallel to the wall, midway between the wall and the fixed point, while the other student, who acted as the curve, thought that it was a semicircle with a radius of 5 foot-paces and its centre located at the fixed point. She was moving to different positions—curve points—with her back to the wall and looking at Mary, repeatedly moving her hands perpendicularly towards

the wall (Figure 18),[142] in order to indicate each curve point's distance from it.

- The other student and the first author, in order to show that the curve could not be a straight line parallel to the wall, used the right triangle formed by the fixed point, the midpoint—located on the straight line segment midway between the fixed point and the wall—and another point on the line parallel to the wall from the midpoint.
- The students could not reach an agreement on what the curve would look like, so they could not conceptualise it and try to identify mathematical relationships; and they chose not to use notes, only their bodies. At some stage the first author suggested that finding more curve points might help them understand what kind of curve it is.

Mary seemed puzzled throughout the activity, and after the break the teacher, who was engaged in another group during the particular investigation, asked what the students thought of it; the other student said "we were trying to find out another point" and Mary responded "we were struggling with it".



[142] The picture is included with the permission of the student.

Figure 19. Mary's "little diagram", when it was completed.

3.4.2.2 The diagrammatic second stage - The empirical 'bird's eye view' intuition

Mary started the second stage of her negotiation at home the same day with a diagram of the fixed point and the 'wall' (Figure 19 shows the same diagram when it was completed). Her diagram has its *intentional origins* in the portrayal of the classroom situation: "when I got home I just drew this little diagram ... I think what I couldn't understand on the day [in the classroom] was that this wall went on forever!" (first interview). The importance of the diagrammatic stage invites us to look closely at how it happened; the original moment was the *empirical intuition* according to Husserl, which emerged and enabled the transition, as Mary described it in her interview:

> Once I could draw a diagram and look back [to the classroom experience], I look at it from *this* point of view; it's much easier than being in *that* point [i.e. the fixed point in the classroom], that looking, like, when standing here [she points to the fixed point on the diagram] *all you see is the wall!* [see figure 19] You don't see all this is going on as well. So once you can see from a *bird's eye view* it's easier to.

Her *intentionality* was fulfilled by intuiting the diagram as a 'bird's eye view' perception of the classroom. Her new perception appeared 'in one stroke', since it was not by induction that she realised that the sought curve could not be a straight line parallel to the wall, as she thought in the classroom; it just did not make sense anymore because she could now actually *see* it, from her 'bird's eye view' perception of the diagram. She also realised 'in one stroke' that her position in the classroom did not signify a straight line but a fixed point. So *immediacy* was an important feature of the transformation of the paper diagram to the classroom re-presentation and the dispelling of her classroom misconceptions. It is an example of the *embodiment of perception,* when the intuitive *moment* is also the *moment of transformation.*

> In order to appreciate the empirical character of this intuition we need to

look closer at its *visual* and *kinesthetic* features, and motility as basic intentionality (Merleau-Ponty, 2002a). As I focused on the intentional origins of the perceptual object (the classroom re-presentation through the diagram) I applied Husserl's theoretical ideas about lived-experience, in my attempt to explore how it came to life in a new sense. The diagram, more than representation of existing knowledge, was an embodied act that "constituted new relationships between the person doing the mathematics and the material world" (DeFreitas & Sinclair, 2012, p. 134). By "responding to the call" of the diagram (Merleau-Ponty, 2002a, pp. 161), kinesthetically and visually, her pre-reflective, *operative intentionality* was directed to the 'bird's eye view' of the classroom setting: It was an *embodied operation* that took place (not a positing thought), that 'I *can* identify the diagram as *such a perspective* of the classroom, here, on the piece of paper'.

We notice how the *operative intentionality* manifests itself through "what would be missing without it" (Merleau-Ponty, 2002b, p. 44, cf. Part 1, §1.4.5): Mary acquired a particular *sense of space,* emanating from an *operation* of a plan perspective on the paper as a single object, which preceded any positing act; because I cannot posit anything before there is space for such a positing, and in particular, space that I can inhabit with my *living body.* It was only *due to* the new perception of the diagram by the operative intentionality that the reflective, positing (intentionality of) *act* could appear, that 'I was standing there, at that point, and this line is the wall'. And it can serve as an example of how the *living body* is present in the perceptual field, as Husserl and Merleau-Ponty have predicted it (Husserl, 1970a, p. 106; also cf. Merleau-Ponty, 2002a, pp. 87-88).

It was not only Mary's diagram but also her classroom experience that acquired a new sense due to her bird's-eye-view intuition, since she could now "look at it from *this* point of view; it's much easier than being in *that* point, when standing all you see is the wall!"; the classroom's yet unexplored experience was re-presented on the piece of paper, ready for the student's mathematical interface to be applied.

3.4.2.3 The formal third stage of the investigation and the intuitions that took place

Mary's intention to mathematise the diagram was confirmed when she directly introduced the Cartesian coordinate system into it, thus entering the formal treatment of her investigation. She placed the fixed point at the origin and the wall became a straight line parallel to the x-axis (Figure 19). Due to her 'bird's eye view' intuition every formal treatment that followed became equivalent to a classroom situation, thus imbuing her abstract ideas with an embodied empirical sense.

3.4.2.3.1 The empirical intuitions of the symmetrical points and the parallel lines

She easily found the first 3 points that belonged to the curve, namely the (10, 0), (-10, 0) and (0, 5). Then she *visually* intuited that the curve will be symmetrical to the y axis and while she was looking for more points, the first key idea in the formal stage of her enquiry appeared. She thought that if she would draw lines parallel to the x-axis and the wall in the area between the x axis and (0, 5), they would intersect the sought curve in 2 points (the grid paper has played a part in her empirical symmetry intuition, since she did not actually draw any lines parallel to the 'wall' and the x axis—see Figure 19):

> So I had this point, this point and this point. so then, I was like 'there must be another point; so what would happen if I'd change, if I moved this point so [that] it was on the line of y=4', yeah? (interview extract)

She recalls a *feeling of certainty* ("I was like 'there must be another point'") and there is already kinesthetic language involved ("what would happen if I'd change, if I moved this point"). Her empirical intuitive idea that the curve is symmetrical to the y- axis and that it cannot go any further than 5 towards the 'wall' made her parallel lines intuition possible. She saw these parallel lines as effective for a curve that was not there, apart from the 3 points that indicated the symmetry. And her feeling of certainty motivated her subsequent actions. The spatiality of her insight (the spatial reorganisation of the space between the x-axis and (0, 5)) and the actual sense of motion as she *"moved* this point" bear evidence of the *kinesthetic*

features of her intuition.

Mary then "continued altering the y coordinates so that the distance from the wall changed" and "pulled out a triangle *in the hope* it will help us calculate what the x coordinate would be in order for the distance from the wall to the origin to remain equal". She involved the Pythagorean Theorem and her hope acquired formal ground by finding the first point (($\sqrt{20}$, 4)), and the whole process became a *technique* as she called it: "The technique seemed to be working so I decided to continue to use it".

The empirical intuition of the lines parallel to the wall with its kinesthetic features, based on the visual intuition that presented the sought curve as symmetrical, reorganised the space between the origin and (0, 5). Her intuitions constituted space within space (since the lines parallel to the wall and the x-axis represented for her their distances from the wall and the x-axis, respectively), and brought objects to the surface of consciousness (lines parallel to the wall and line segments perpendicular to the wall). Culturally mediated concepts (such as the coordinates, the grid paper and the Pythagorean Theorem) interplayed with kinesthetic and visual personal experience in order to constitute the new object (the parallel lines as the way to fix the x coordinate) and the new tool/technique that realised[143] her new object.

3.4.2.3.2 The intuitions related to the x^2 curves

Mary had found the first 5 points of the curve ((0, 5), (10, 0), (-10, 0), ($\sqrt{20}$, 4), ($-\sqrt{20}$, 4)). According to Merleau-Ponty (2002a) attention "first of all presupposes a transformation of the mental field, a new way for consciousness to be present to its objects" (p. 33). The 5 points to which the student directed her attention to, became a *figure* on the *ground* of the paper and the other lines she had drawn. As she clarified in her second interview, "it was an aha-moment, I noticed that all of

[143] The application of her new object in her technique is also perceived as a better *apprehension* of the newly-constituted object (cf. the theoretical section §5.1).

the points were going in the same fashion and thus there must be *a rule* as to what they were all following". A curve connecting these 5 points appeared in her consciousness. After she "realised that there was some pattern" she intuited that "this shape of graph", which she had just intuited out of the 5 points "is resemblant [sic] of the x squared graph". That is how her double intuition occurred: at first she noticed that there is "some pattern" between the 5 points and then the idea of "the x squared graph" appeared in her consciousness.

The extraction of a form out of the 5 points could be explicated as an *empirical* intuition according to Husserl and the result was the creation of an object, a shape that was brought to consciousness. The *immediacy* of intuition concerns the moment that the intentional act was fulfilled, the moment when sense was attributed to the intuitive *operative* intentional act, by the separation of the form out of the 5 points. The form of the 5 points has a pre-linguistic visual expression, as it transforms the real objects (the 5 points) to a new kind of object (a *profile* of the sought curve).

The second intuition associated the object that came out of the first one, with the x curves. It bestowed the image of the first intuition with an abstract categorial feature, i.e. as belonging to the family of the x^2 curves, which is what makes this intuition an *abstract* (categorial) intuition. It is a moment in which we feel how the intuitive contact took place: the student's *operative* intentionality extracted the form out of the 5 points and her *intentionality of act* (to find "a *rule*") posited the form as an x curve figure; It was "proper learning" (as she called it in the interview) that helped her to find an answer but it was only intuition that allowed it to appear in consciousness at the right moment.

3.4.2.3.3 Integration of processes - the constitution of the first formula

Mary objectified the set of arbitrarily chosen lines parallel to the wall and their corresponding perpendicular line segments (parallel to each other) and she used it as a *technique,* by applying it to the generation of another pair of curve points ($(\sqrt{40}, 3)$, $(-\sqrt{40}, 3)$). She then wondered if there was "a point that would be of equal distance to the wall and the origin ... once the *moveable point* passed the x-

axis". By application of her already objectified technique she overstepped her initial intuitive conviction that the curve is limited by the x axis... It is a case that her formal treatment corrected the pitfall of her visual empirical intuition, showing that intuitions may be misleading but they are not necessarily end barriers; and it exemplifies how intuitive mathematics in action is legitimate mathematics.

At this point Mary employed instruction based arguments, without parting from her intuitive approach. She noticed that the x curve is translated to the maximum point (0, 5), so she thought of the formula $y = 5 - x^2$. She tried the point (10, 0) in the formula but she got -5 = -100. And then she manipulated the false arithmetic relation into the correct formula ($y = 5-\frac{x^2}{20}$), by using the point (10, 0) and focusing on number 10, as the following coursework extract shows:

$20(5) = 100$ [Starting with an arithmetic relation that corrects the error]

$5 = \frac{100}{20}$ [Focusing on 100—she draws 20 with it in the second term of the equation]

$0 = 5 - \frac{100}{20}$ [Introducing 0 – intention to use (10,0) as the entry point to the formula]

$0 = 5 - \frac{10^2}{20}$ [Intention to use 10 – focusing on 10]

$0 = 5 - \frac{x^2}{20}$ [x replaces 10]

$y = 5 - \frac{x^2}{20}$ [y replaces 0]

This is how Mary constructed the formula $y = 5 - \frac{x^2}{20}$. Her intention to use (10,0) because of 10 (i.e. the distance between the wall and the fixed point) is indicated by prioritising the substitution of 10 by x over the substitution of 0 by y.

Mary then tested the point ($\sqrt{20}$, 4) in her formula and she was convinced that the formula is correct. Two points ((10, 0) and ($\sqrt{20}$, 4)) were enough to constitute the sense of generality for her, which was something repeated later on (see the following section). The introduction of the formula by the student became an objectification of the coordinates of the points having the property of 10 units' equidistance from a straight line and a fixed point. Finally, her persistence in

number 10 throughout the process became the preparatory step to her last intuition, which brought the general formula to the surface.

3.4.2.3.4 The intuition of essences related to the general formula

There is a particular kind of abstract intuition, which is a *categorial* intuition that Husserl calls *intuition of essences* and it is characterised by apodictic evidence (Lohmar in Haritmo, 2010, p. 74). The example of Mary's last intuition will allow us to get a glimpse at the main features of this kind of intuition, which holds particular significance for generalisations and mathematical proofs.

Although Mary had answered the particular mathematical enquiry, her intuition pushed her investigation further, by objectifying the whole process into a *new tool* in order to *extend* the task. The distance between the wall and the fixed point remained her intended number as she repeated the whole process and found the formula of the curve when the distance between the fixed point and the wall is 6 units instead of 10 $\left(y = 3 - \frac{x^2}{12}\right)$. As she explained in the interview she had already seen the essential relations between 10 and 5, 20 in the first formula before she figured out the second formula. But she became *increasingly* certain concerning the part of number 10 in the first formula, as she applied identical processes to the second formula. And she acquired the formula $y = \frac{a}{2} - \frac{x^2}{2a}$ of all the curves with the same property, by replacing 10 by a variable in her initial formula.

The particular qualities of the *intuitions of essences* surface here, such as the three 'moments' that conjoin, in the following sequence:

- The first formula is the *starting example* (Lohmar in Haritmo, 2010, p. 78), the *object of experience* (ibid., p. 83) and it became *the leading example*, already incorporating essential embodied elements as the previous analysis has shown.
- *The generality that the leading example gains through the modeling of the whole process* in another, intended example (the second curve). The matching of the first formula to the second one (and the correspondence between 10 and 6) was decisive for the process of its generalisation.

- *The "synthesis of coincidence"* (ibid. p. 79), through *the recognition of number 10 as such,* stripped of its particular originations and seen as a 'variable number'. This operation is instantiated in the leading example, by pinning 10 down to the mathematical symbolic frame and transforming it to a variable. That is how number 10 was recognized as essential for the expansion of the formula and finally acquired a new quality.

The 3rd moment is the conclusion and the core of the intuition, since all steps were realised in a stroke, in the recapitulating act of coincidence (the third 'moment'), typical of the intuitive *immediacy*. Mary sidestepped the various connotations of number 10 (sensory, algebraic, geometric) and identified it as a number—as 'any' number. What made this possible is what Lohmar calls the interpretation of the "special synthesis of coincidence" (Lohmar, 2010, pp. 79, 80; also cf. Husserl, 1973, Section 1), based on the singular example of the first formula and the generality it acquired from its reflection on the other formula and curve through number 10. The *'synthesis of coincidence* concerns the *qualitative transformation* - through a novel *thematical identification* - of the particular, out of the awareness of the generality. The synthesis of coincidence occurs between the subdividing acts - the two formulas. It is the apprehension of the coincidence as the identity of the general feature that becomes the *essence,* expressed as the general formula - the intended object of a *new category.*

3.4.2.4 The curve as a "moveable point" – Coexistence of abstract and empirical intuitions as a sign of deep understanding

The phenomenological analysis allows us to see the linkages between the empirical and the abstract (categorial) intuitions, and the transition from the pre-reflective to the reflective field of experience.

The student perceived the sought curve as a "moveable point", which is an expression she repeatedly used for the curve in her coursework. Her sensorimotor expressions ("moveable"—see Mary's quotation in Part 3 § 3.4.2.3.3) are signs of how she explored the relationship between the locus of the parabola points and the functional one. She vividly described it when she had accomplished the task:

> Like for that I am positive that I can say that if this is true and if this part was *a,* you could stand on anywhere on the *line* $y = \frac{a}{2} - \frac{x^2}{2a}$ and you would be equidistant from the wall and a fixed point. (first interview extract)

So the formula could be rolled back to its empirical origins, integrated in a perception that permeated the student's investigation: it's always points, either points to be traced with lines parallel to the wall, points to judge if the formula is the correct one, to manifest the generality achieved through the concrete particular case, and so forth. It is in this sense that Mary's reactivation of the parabola was in its turn sedimented to a line that "you could stand on anywhere" and "be equidistant from the wall and a fixed point". Adding a new perception to the formulas of x^2 functions, in her broader mathematical interface.

Finally, what is remarkable and pertinent to this research is that due to a sequence of successful intuitions she constructed formulas that *she only later knew that they are called parabolas,* and a general formula, in a bottom-up process. It was this lack of names and ready-made information often confused with knowledge that enabled her to realise the parabolas, through their equidistance property.

PART 4 – Discussion and Conclusions

Introduction – a summary of what has been achieved up to this moment

In this section I will introduce the main themes of this chapter through a concise description of how the previous chapters have contributed to the argument employed in my research, and how they have paved the discussion that follows.

In the theoretical section I delineated the complexity of the cognitive lived-experience, and the significance of a neglected side of the learner's cognitive praxis, the one related to the *pre-reflective* or *passive stage,* within which the *operative intentionality* synthesises object-like entities that are intentionally driven towards objectification (see the Table 1 and the Graph 3 in Part 1 §1.5.7). And I haven't been engaged so much with the most easily recognisable *intentionality of act,* which is the positing intentionality that constitutes objects, thus endorsing them to the discursive field and ultimately to the significative field of mathematics. The latter intentionality was analysed and linked to the former in the analysis of the data that followed, in all three investigations that came under my Husserlian/Merleau-Pontyian analytical lens.

I chose as most appropriate for my theoretical and methodological purposes the first-person perspective, i.e. the perspective of each of the three students that was engaged in the investigation each time, in the classroom or at home. For this perspective I employed Husserl's transcendental phenomenological reduction, which I introduced as the necessary phenomenological methodology, in the theory and the methodology/methods sections. Due to the reduction I bracketed influences of cultural tools and practices, in order to investigate the end-points of phenomenological reductions, which are the intuitions for Husserl (Hintikka, 2003, pp. 178, 179).

In order to access the abstract intuitions in particular, the phenomenological reductions were the only way, according to the theory.[144] Due to my

[144] "[T]he intuitions that are not already given to me in sense perception are uncovered by phenomenological reductions" (Hintikka, 2003, p. 178).

phenomenological attitude and the reductions, informed by Merleau-Ponty's radical reflection, the data analysis recognised two major kinds of intuitions, namely *empirical* and *abstract* intuitions, and the analysis manifested their grounding on the learners' *lived-experience*. Moreover, various links between these two kinds of intuitions were detected and analysed.

I applied a close analytical look at the effects of the NoM sessions on three very different cases, and I tried to clarify my perception of the radical orientation of Husserl's mature cognitive approach, by emphasising the intentional and the horizonal aspects of the objects' emergence, in each case. In fact, the *emergence* of the object—as an object-like entity, followed by the constitution of the latter as an object—was at the centre of my investigation. I distinguish between the appearance of the object-like entity and the object itself (the object's *constitutive 'moment'),* since these two 'moments' can be very close but they are not identical.[145] Moreover, these two moments are important for the clarification of the phenomenon of *objectification,* since they bring to the surface the essential role of the two intentionalities that correspond to these two 'moments', namely the *operative intentionality* and the *intentionality of act*. The data became findings as they were lit by the theory, while they clarified the theoretical ideas involved, and they exemplified how subtle theoretical notions may appear in 'everyday' learning lived experiences. As Husserl (1983a, p. 41) put it:

> Evident data are patient; they let the theories pass them by, but remain what they are. It is the business of theories to conform to the data, and the business of theories of knowledge to distinguish fundamental kinds of data and describe such kinds with respect to their proper essences.

[145] An example of this *différance,* concerning the emerged object-like entity and its constitution as an object (which was the "optimal solution" for Ivan), is exemplified in the following section 1.3.2.

4.1 Ivan – A mature intuitive approach on mathematics that was released due to the special character of the NoM course

4.1.1 Preliminary discussion of the data analysis key issues

The main finding in Ivan's case was his *intuition of essences* and the concomitant *synthesis of coincidence,* performed during his investigation of the 'New York cop' task, which combined spatial, visual and algebraic features. In the analysis section we followed Ivan's investigation of the task, starting from his activity during the session with his group, which took place without interference by my presence and without any *specific* direction or guidance on behalf of the teacher. The teacher gave general suggestions, such as *not* to prioritise finding formulas but to use "an *intelligence* about *moving* around the *space,* and *seeing* how you can *cover it*". He also asked the students to adopt a reflective attitude towards their activities and their group activity, by becoming aware of "the *systematicity* of the ways" that they would locate the formulas (section 3.2.1 in the data analysis). And, deeper than his "general suggestions", the teacher had his own intentions for the introduction of his investigations, while I was in the process of collecting data, before they became findings. It would be very useful to take a closer look to the teacher's intentions, since it will help us with discussing the analysed data.

4.1.2 Important issues accruing from the data analysis and related to Ivan's synthesis of coincidence

I intend in this section to discuss deeper the principal findings of Ivan's case, starting from the *synthesis of coincidence* that enabled the emergence of the *intuition of essences,* since they were disclosed as the peak moment(s) of Ivan's investigation. By unfolding the data that I collected I showed how the essence of the 'diagonal pattern'—namely the object that was intuitively constituted as the intended object—drew its *intentional origins* from Ivan's intention to find a *holistic answer,* namely a pattern that would deal with *all* cases. There are important issues that the data analysis chapter supports but does not mark, does not make explicit. Such issues, which occur 'naturally' from the analysis of the data and the phenomenological

attitude that investigates and evaluates them, will be explored in this section. Three arguments, concerning the *synthesis of coincidence* that realised Ivan's *intuition of essences,* will be disclosed, and show how a common event can come freely up to life; while the tools that we were afforded in our theoretical and analytical journey will be expected to bear fruits equal to the effort that was undertaken. Why is it a common event? Because it summarises a *moment* that is common to all mathematics classes: *the transition from the concrete to the abstract mathematical object,* whatever that may be. Moreover, it exemplifies *a process of generalisation,* with a detailed organisation of natural numbers in matrixes (grids), according to the number of rows and columns. We need not be misled by Ivan's *readiness* and 'natural intuitivity; Ivan's case is exemplary because it makes crystal-clear how the transition is done after struggle and little steps, which join in a "breakthrough moment". The 'vertical lines' cops idea that Carry had suggested and Ivan had adopted was one of the ways that Ivan tried to fix a relation between the square grids (see the numbers and shapes around each square grid in figure 3 of Part 3 §3.2.2.1, in the data analysis). But it became particularly significant when he applied it to the $3 \times n$ grids. What we followed with Ivan was a compound learning experience that its complexity reveals its *authentic speech,* in Merleau-Ponty's terms. Our understanding of the theatre of patterns that Ivan went through until the session was finished,[146] up to the moment that he spotted the *diagonal pattern,* derives from our understanding of the significance of the *operational* potential of his embodied consciousness, which *prepared* the ground and *made possible* the final synthesis for Ivan. In discussing the data analysis findings it will be useful to take a closer look at the principal features of Ivan's intuition.

[146] I remember vividly how many papers Ivan left behind him, full of drawings, without giving them to me in order to photocopy them. His face full of discontent due to the lack of results, as he crumpled some of these papers.

4.1.2.1 The qualitative transformation of the particular, out of the awareness of generality

The synthesis of coincidence was exemplified as the final moment in a three-step process: it started from the *recognition* of the *starting* example, it continued with the *variation* of the *starting example*, and it culminated at the *synthesis of coincidence* that recognised the 'leading pattern' in what first appeared in the *starting* example and even in the 3×3, 4×4 grids alone (Figure 5 in Part 3 §3.2.1 of the data analysis), which had first triggered his attention. A *novel thematisation* of the pattern that was 'isolated' in the 3×3, 4×4 grids as the *leading* pattern that would cover all cases was achieved, due to the *variation* of the *starting example*, that is the modelling of the six square grids on the six 3×n, grids.

As it was stated in the analysis of Ivan's 'synthesis of coincidence', the latter was concerned with *the qualitative transformation of the particular, out of the awareness of generality* (Part 3 §3.2.3.1). It is obviously crucial for the broader discussion that a theoretical and practical elucidation of this "*awareness of generality*", and a corresponding explication of the "*qualitative transformation of the particular*" are needed.

The *particular* was his realisation that an 'asymmetrical' pattern gave the best results in the cases of the 3×3, 4×4 grids; we noticed how he intentionally focused on it when he compared it with the symmetrical patterns of the 3×3, 4×4 grids, by drawing them next to each other (Part 3 §3.2.2.2). The *transformation of the particular to the general case*[147] was achieved due to the intuitive *synthesis of coincidence,* which was the result of the application of the 'asymmetrical' pattern to square and 3×n grids. And *the generality was achieved* by the constitution of the 'diagonal pattern'[148] as the pattern that would be applied for all grids.

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[147] This 'moment' is interpreted as the awareness of the generality (the abstraction) in the particular, by CHAT, in the sense that the concrete case is endowed with more 'knots' of meaning (and consequently of signification), thus entering (re-presenting) the realm of cultural, pre-established epistemic concepts/objects.

[148] The expressions *'diagonal pattern'* and *'asymmetrical' pattern* refer to the same pattern, which was the one that Ivan used in order to solve the 'New York cop' task.

It needs to be added that such a constitution would not bring any significant results if it would not be integrated to Ivan's previous explorations, which brought two important spatial conceptualisations:

- Ivan's notion of the *"inside cops",* which led him to the configuration that was based on starting with 2 cops on the outside, which both had a 2 blocks distance from the top left corner of his drawings (Figure 2).
- Carry's notion of the *'vertical lines' cops* (the amount of cops for each vertical line of the grid), which was adopted by Ivan (cf. the numbers under the 3×8 grid in Figure 6).

The first notion was the key in finding the 'asymmetrical' pattern itself, since it determined the form that is common to all grids, i.e. the configuration of cops at the top left corner. And it became part and parcel of Ivan's conception of this pattern, thus integrating his best results from his exploration on the square grids. The second notion allowed the discovery of the arithmetic pattern (1, 1, 2, 1, 1, 2, 1, 1, 2), which became the basis for the algebraic treatment that followed. In the next section we will explore how the transformation of the particular to the general case was realised by Ivan.

4.1.2.2 Operative intentionality and the object, before constitution – the building up of the feeling of certainty

The intuition of the 'diagonal pattern' unlocked the task for Ivan for one more reason, since it gave him the feeling of certainty that he had found the appropriate pattern, which allowed him to move *methodically* beyond the square grids.

But let us examine more carefully Ivan's *first* step of his *synthesis of coincidence:* he checked the table that he had drew for the square grids, and he did not find any link between the numbers of cops in that table;[149] he then did not question the asymmetrical diagonal pattern that he was using, but he tested the same pattern for the $3 \times n$ grids instead. We notice that even before the second step of his *synthesis of coincidence* there is a feeling of certainty that there is already

[149] Ivan mentions in his coursework that "it seemed like there wasn't any logical way forward".

'something' to be explored further. The pattern as it appeared in the $3 \times 3, 4 \times 4$ grids is an *object-like entity* for Ivan, since it has not yet become the "diagonal pattern", the "optimal solution". As soon as it was isolated and compared with another (symmetrical) pattern it already became something that he wants to determine closer, to apply it again to a second set of grids, namely the $3 \times n$ grids. Therefore we can detect the manifestations of *operative intentionality:*

- In synthesising the 3×3 and 4×4 grids with the asymmetrical pattern, as *one* pattern, but not yet as an object (a synthesis of identification).
- In sustaining the pattern of the $3 \times 3, 4 \times 4$ grids in the square grids, and
- In driving a new application of the same pattern to the $3 \times n$ grids.

Thus we testify how the operative intentionality operates, and consequently how the "emergence of the sense" (Merleau-Ponty, 2002b, p. 18) takes place, before the object is constituted by the *intentionality of act,* as the "optimal solution". Moreover, we see that already at the first step of his *synthesis of coincidence* Ivan started building up the feeling of certainty that he had found a suitable pattern for all grids. We could also say that the pole of symmetry (Kolen, 2005) concerning the intended object, was *instituted* through an embodied *operation* that brought together previously neglected possibilities. In terms of Ivan's *intentional* approach, the *intentional horizons* for the sought object that was about to emerge were rearranged, and his *protentions* concerning square grids and symmetry shifted to "empty indications"[150] involving orthogonal grids and the diagonal pattern. The inductive approach that I used, which 'predicts' (anticipates) the presence of operative intentionality and the object-like entity in particular areas of the learner's non-discursive cognitive operations, although discovered in analytical practice it is already predicted by Husserl (Husserl, 1970a, p. 51):

> In the most primitive way, even the ontic certainty of any straightforward experience is inductive. Things 'seen' are always more than what we 'really

[150] *Protentions* and *empty indications* are two terms that refer to the same thing, i.e. to the anticipations/expectations that are an integral part of the learner's embodied conscious acts in her present. For the importance of protentions/empty indications for the prefiguring of the sought object, see section 3.3 in the theory chapter.

> and actually' see of them. Seeing, perceiving, is essentially having-something-itself *[Selbsthaben]* and at the same time having-something-in-advance *[Vor-haben],* meaning-something-in-advance *[Vor-meinen].* All praxis, with its projects *[Vorhaben],* involves inductions.

Since we deal here with a "phenomenology of the experience"[151] (Derrida in Husserl, 1936, p. 50), my detailed exploration of

- the *object-like* entity, which appeared to and was spotted as worthy for exploration by Ivan—and
- the operative intentionality that synthesised it as such, and pushed it forward,

became possible due to a methodological reduction that sought for the intending and the intended, on the ground of Ivan's lived experience, right from the start, parenthesising all that would blur the actual phenomenon. Hence Ivan's history of objectification becomes instructive of a path, which might be more common than one thinks, since it is related to students that acquire the freedom to act according to their intuitions, and the concomitant mathematical self-confidence. Moreover, it sets an example of how this self-confidence is built, as it is based on practicing intuitive investigations, as is the case with Ivan's natural-like intuitivity, to which the analysed episode is but a mere example.

The analysis of Ivan's learning episode shows

- The anticipatory force of the phenomenological methodological instruments that I have used throughout the study, such as the phenomenological reduction, the epochè (bracketing) and Merleau-Ponty's radical reflection, which enabled this detailed account.
- The anticipatory, "inductive" potentiability (potential and ability) of Ivan, from the pattern that emerged from the $3 \times 3, 4 \times 4$ grids, to square and $3 \times n$ grids, and then to a pattern and nine formulas.

[151] Derrida (ibid.) writes: "A phenomenology of the experience is possible thanks to a reduction and to an appropriate de-sedimentation".

Revisiting the empirical texture of Ivan's investigation we notice that the new concept of the *grids with the same height* replaced his (exclusive) focus on the *square* grids and led him to the construction of nine final formulas (Figure 9 in Part 3 §3.2.3.2), thus covering all space. But the analysis showed—and Ivan's interview confirmed [152] —that "the shape pattern came first", that it "was the main breakthrough" (Ivan's interview extracts). The arithmetic pattern that emerged from the $3 \times n$ grids was *additional* to the *synthesis of coincidence,* which constituted the "diagonals" as the appropriate, generalisable pattern;[153] and it *sealed* the *intuition of the "diagonals"* as *the source of suitable conceptualisations,* leading to the *closer determination* of the "diagonal pattern" (the new object) through the task (see Graph 3 in Part 1 §1.5.7). And the *closer determination* of the intuitive object of the "diagonal pattern" commenced with the *specification*[154] of the "diagonal pattern", as the functioning—i.e. algebraically effective—generalisable use, on *grids with the same height*.

Another student (Diana) presented a few formulas in her coursework (see appendix) from the work that was done in her group during the *New York cop* session. Diana also mentions in her interview (section 3.6 in the analysis chapter) the 1, 1, 2, 1, 1, 2, 1, 1, 2 arithmetic pattern that Ivan found at home, which she did not consider as significant for her approach, since she was not aiming at a consistent pattern for *all* grids. I did not further analyse Diana's investigation of the *Doubling modulo,* since the result that is pertinent to Ivan's investigation is already apparent:

Structure does not just appear, unless supported by the intentionality that yields it visible for the cognising subject.

There is also something less apparent yet significant from the theoretical viewpoint of my research, concerning the *genesis* of the object:

Intentionality and its object do not co-emerge.

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[152] Part 2 § 3.2.3.

[153] The diagonal pattern worked, i.e. it achieved the minimum amount of cops for 12 grids, 6 square and six orthogonal ($3 \times n$) grids, *before* the arithmetic pattern surfaced.

[154] See *specifying fulfilment* in the Graph 3 of Part 1 §1.5.7.

Intentionality precedes the object's constitution, and it is manifested in various ways. More precisely, the *operative* intentionality synthesises the object-like entity (that will become the constituted object), and then the object is constituted, as it co-emerges with the *intentionality of act.*[155] This process was clearly manifested in this section, in the analysis of Ivan's first and second step of his synthesis of coincidence.

4.1.2.3 Ivan's case under the light of Diana's treatment of the 'New York cops' task

The data analysis revealed the fervent activity in the group of Ivan, Carry and Donald, who did not manage to resolve the different approaches that Ivan and Carry developed throughout the session. The primary target of the data analysis was to lay bare the operation of the *intuition of essences,* through the three steps of the *synthesis of coincidence,* and the impact of Ivan's intuition on his algebraic expansion of the task. The target here is to get into these genetic moments from a practical point of view, by showing that these theoretical notions actually refer to moments that occur far more often than we would expect, and are recognisable due to a particular analytical operation that I exemplified in the previous section, and which Merleau-Ponty (2002b, p. 44) circumscribes in his notes on Husserl's Origin of geometry:

> It is the thought of the intentionality which is proper to language, interrogation of language, the *epoche* of language, the suspension (limit of eidetic variation) revealing only the operation, *a contrario.* We do not see the operation, since it operates. We see what would be missing without it; we circumscribe it as what makes speech be a 'speaking of' ... and not be the conscious having of the idealities implied in speech. (emphasis in the original)

As it happens in every class, there were students who did and students who did not or partially did find answers to the New York cop short-sighted policemen task.

[155] In other words, the two intentionalities are not regarded as strictly separated, but rather as dialectically interrelated, through the transcendental subject.

teacher's intentions towards the given task. It was shown in the analysis of the data how:

- Donald merely followed the methods of Ivan and Carry.
- Carry followed the teacher's suggestions by not aiming at finding formulas at all (see Carry's interview during the session).
- Ivan also focused on the shapes of the cops' configurations in the square grids, aiming at a consistent pattern for *all* grids before being engaged in finding formulas for particular kinds of grids.

It was also shown how Ivan actually and literally substantiated the teacher's suggestions (section 3.2.5). Ivan's remarkable intuition of the diagonal pattern brought him to the exploration of the 3×n, 4×n, 5×n, 6×n, 7×n, 8×n grids, and to his new intuition of different formulas, according to the remainder of the columns' number division by 3. Even this latter intuition,[156] how was it initiated? The 3×n grids (six of them) that Ivan drew, and the rhythm of 1, 1, 2, 1, 1, 2, 1, 1, 2 (see Figure 1), already provide a richness of indications towards number 3, which Ivan had at his disposal, and he was focusing on at that moment, even before he collected suitable data to support an argument. We know that the *lived body* is triggered by the allure of the objects that it encounters each moment of cognitive conscious life, and we understand that:

- Objectification is possible due to a synthesising intentionality that operates with association, and separates forms from the learner's lived experience.
- The objects' features, such as rhythm or the shape of dots on a grid do not appear as stimuli, but rather as Gestalts, as *figures on the ground*.
- These *figures* are brought to life due to embodied intentional operations, *before* they are constituted as objects.[157]

We know from Diana's interview (section 3.6 in the analysis chapter) that she and her group had arrived at the same arithmetic pattern that Ivan found. And in Diana's

[156] This intuition is not analysed in the data analysis section, as I chose to focus on the far richer intuition of essences.

[157] See the previous subsection (§4.1.3.2).

coursework (see appendix) we find the same arithmetic patterns that Ivan found. But the difference between Ivan's use of them and Diana's group cooperative work when they confronted them, equates from my phenomenological perspective to Ivan's *holistic intentionality,* which allowed (or 'forced') him to *intuit* that "I could break it up into 3 sequences" (extract from Ivan's coursework, in section 2.3.6 of the analysis), and deal with the grids' width *as three classes,* according to their remainder of their division with number three (known as mod3 classes). This holistic view afforded Ivan to take advantage of the same arithmetic results that we find in Diana's tables: as soon as his *intentionality of act* constituted the diagonal pattern as the overarching pattern for every grid, it was easy for Ivan to treat the arithmetic patterns as particular cases (parts) of the same whole, as the following extract from Ivan's coursework indicates:

> Starting with a 3x2 grid, (3 up 2 along), then the next grid in the sequence was 3x5, then 3x8... which gave me a pattern of 4,8,12,.... This pattern I called "when, (n+1)/3, is an integer".

4.1.4 Conclusion

The analysis of Ivan's intuition has shown that the object of the "diagonals" configuration was in fact *given* in experience, and how the Husserlian approach to experience is in fact distinct from the corresponding Kantian and Cartesian approaches. The object *became known* as it was separated from his lived experience, rather than *constructed* within it. The description of Ivan's activity in the classroom and at home, and the analysis that followed, also attempted to delineate the route from a level of experience that is not subjected to the objective categories, to the reflective, positing experience of an intuitive object at hand.

Ivan shared and adopted ideas during the session, he made several unsuccessful attempts to reach a pattern that could be generalised and finally, at home, he realised with the immediacy of intuition that one of the patterns he had already found was "the optimal solution,… the best way to arrange" the cops. The incubation of his gut feeling throughout the session that there is a unified pattern, paid off in the three steps that led to the *synthesis of coincidence.* The *essence* of

his "diagonals" was that they could be used as the optimal way of covering the grid space with short-sighted policemen, as they could be mathematised.

Ivan constituted his subjective comprehension of the grid space at the moment that he secured its intersubjective—mathematical—dimension. Moreover, the analysis showed how his constitution of the new object (the "diagonals") was a transition from the pre-reflective, operative recognition of the 'asymmetrical' pattern as such,[158] to the visual, reflective and positing intentionality of the arithmetic pattern;[159] this transition is an example of how the *operative intentionality* makes possible the *intentionality of act.*

Finally, the complexity of the intuitive moment analysed in my research serves as evidence of how the Husserlian approach to intuition considers consciousness as *essentially* embodied, and *intersubjective,* being the target and the constituting moment, through the establishment of relations that render qualities communicable.

4.2. Diana – the transformation of negative to positive feelings towards the course, as soon as she managed to transform raw empirical material to abstract mathematical objects

4.2.1 Implications for the student – Conclusion

The 'doubling modulo' activity and its treatment by the student had major implications for her, as the following extracts from her interview manifest:

> it was very very long-winded and it took me *a lot* longer to understand it than it did for everyone else. But now *I've done this* and *it's my way,* nobody else seems *to have done that way*. … I like my way! It's a long way, but I like my way … I think when I'm teaching *I need to get over the urge to give answers… not straight away.* … [A] lot of the time –and it's not just with maths, it's with other subjects– I've been very cautious when I found an answer, *if my answer isn't the same as everybody else's* I've just assumed

[158] It is where Husserl's and Merleau-Ponty's *operative intentionality* is at work.
[159] The *intentionality of act* is at work.

mine is wrong! Because it's not the same as somebody else's … I think I've possibly become a bit more confident with what I'm doing; because *I'm thinking "well, because I've got a different answer to somebody else it doesn't necessarily mean that it's wrong, it just means I thought about it in a different way"*. So it has, yeah it's been good for me! … I think I have realised from these lessons that *it helps people more if you help with their understanding of it rather than just give them an answer straight away; and naturally make them think about it and what they are doing and why they are doing it,* which obviously is very important with wanting to be a teacher. (bold indicating her emphasis, italics added)

Starting from an empirical intuition and due to the character of the sessions that allowed an open frame of what a 'correct' result is, the prospective teacher developed her ideas, devised a method, and cashed in the whole operation to a new perception of learning and teaching. She *separated* the question/answer sequences from her experience in her empirical intuition, which enabled the abstract intuition or intuition of essences (*Wesensschau),* where a new separation from the new layer of data took place, namely the separation of the algorithmic abstract essence from the particular patterns concerning 15 numbers (the hyle, or raw data in Husserl's terms), spread in several pages. Hintikka (1995, p. 183) explains:

> According to Husserl, we can come to know essences by means of empirical experience by separating them from the hyle in *Wesensschau.* The main difference as compared with Kant is that we do not create the essences, we merely come to know them by separating them from the sensory data.

As the aforementioned extracts and further observation of the prospective teacher in this course as well as in other courses showed, she reconsidered her approach to learning and teaching in two major issues, as her "frustration" was radically transformed:

- Her results and methods can be valid although different than the other students' results and methods.

- Teaching and learning is not about giving or getting the answers "straight away", but rather about "help[ing] [people] with their understanding" and "making" the students "think about it and *what* they are doing and *why* they are doing it".

The analysis showed how new objects surface through empirical and abstract intuitions, and the interrelations between the two intuitive stages (empirical and abstract). Furthermore, we noticed how the materials of intuition are *neither internal nor external,* as there is an *overlap* of the objects as they are given:

- In their *material form*—either drawn directly from experience, or as structured *questions and answers* concerning numbers.
- And their *intended form* that intuition separates from the sensory or abstract data (respectively).

"It is crucially important to emphasize that, according to Husserl, there is an actual interface of my consciousness and reality, that reality in fact impinges directly on my consciousness" (Hintikka 1995, p. 83). The materials of intuition are not prepackaged, but rather it is *reality impinging on consciousness* in the form of unstructured raw materials (Hintikka 1995, pp. 102-103). The questions, the answers and the judgments were transformed to commands within algorithms, which in turn formed a flowchart, in which the value of the entrance is any given number. This is how we have access to reality and not due to some predetermined internal mind structure. The mind-body dualism is cancelled, since consciousness for Husserl is embodied (Tito, p. 186); and the relation between consciousness and reality is *mediated* by *intuition.* The phenomenological analysis can investigate empirical or abstract objects as they are intended by the learner, which *become known* in her lived experience by separating them from the unstructured raw material, rather than by been *constructed* in experience through our already categorial intelligence.

4.3. Mary – The representative case of all detected intuitions and objectifications

4.3.1 Classification of intuitive expressions, as they were detected in Mary's activity

Although all the intuitions found in Mary's treatment of the parabola task are analysed with the Husserlian classification of *empirical* (*Anschauung*) and *abstract* intuitions *(categorial-Wesensschau)*, a classification according to their different intuitive *expressions* follows, with their relations to the classifications from the literature review and some corresponding examples from the findings. It is rather a *grouping* of the intuitive *expressions* that were detected, under the four headings that follow, aiming at a *mapping* of the *embodied* features that were also decisive for the realisation of *all* intuitions. In other words, it is not an empiricist approach that leads us here to the aforementioned grouping, but a deployment of the intuitive features that could allow a question to be posed and be given a preliminary answer; that is, which is the ground on which the intuitive expressions and their corresponding intuitions are emerging from, as they yield objects and objectifications anew. Hence we discern what it is precisely that makes the resulting objects and objectifications possible; the answer being that it is no other than the *lived body,* the learner's *embodied consciousness*. A first task is thus to map the area of embodied expressions not with an empiricist eye but with a *transcendental* descriptive eye; ready to extend it to the results of our observations, applying a new reduction and suspension of judgment towards the *object-as-it-is-intended* by the transcendental ego. This is precisely what the phenomenological immersion to each phenomenon is all about. It is to sense the *horizonal/intentional lived-world* of the learner(s), where objects and objectifications appear as new, and they definitely are linked to the real, for Husserl's strong realism (e.g. principle of all principles). Having said that I ought to add for a fuller view that Husserl has been the theoretical explicit or implicit reference to many of the founders of the post-modernist tradition (Ricoeur, Derrida, Merleau-Ponty, etc.).Therefore, the classification that follows is in rough but crucial terms a first mapping of the operations of the *lived body:*

- ***Informal logic* intuitive expressions:** they are *empirical* intuitions in Husserl's terms, and they are often holistic and plausible intuitive expressions, and/or relying on a physical model or on some leading examples (cf. Poincare's 1st and 2nd categories and Davis & Hersh's 3rd, 4th and 5th categories).
    - Examples of these expressions concern Mary's certainty that the curve is symmetrical to the y axis when she had found just 3 points, which led her to the intuition of the lines parallel to the wall (§ 3.4.2.3.1). Another example is related to seeing the x-axis in her drawing as a barrier for the curve (§ 3.4.2.3.3.), before her other intuitively based work allowed her to push on to a more mathematically complete view. Both examples indicate immanent kinesthetic features, thus bringing evidence for the embodied roots of what we call 'informal logic'.
- ***Kinesthetic* intuitive expressions:** they are *empirical* intuitions in Husserl's terms, similar to the visual intuitive expressions but based on sensory-motor experience (cf. Poincare's 1st category, Fischbein's primary intuitions and Davis & Hersh's 4th and 6th categories).
    - Examples of these expressions are the kinesthetic aspects of the 'bird's eye view' intuition (§ 3.4.2.2.) and the intuition of the lines parallel to the wall (§ 3.4.2.3.1.).
- ***Visual* intuitive expressions:** they are *empirical* or *abstract* intuitions in Husserl's terms, based on vision, occurring from adequate instruction and/or personal experience (cf. Poincare's 1st category and Davis & Hersh's 2nd category).
    - Examples of these expressions are the visual aspects of the 'bird's eye view' intuition (§ 3.4.2.2.) and the intuitions related to the x^2 curve (§ 3.4.2.3.2.).
- ***Mathematical–formal logic* intuitive expressions:** they are *abstract* intuitions in Husserl's terms, formed after adequate instruction and able to create mathematical reasoning (cf. Poincare's 3rd category, Fischbein's secondary intuitions and Davis & Hersh's 6th category). This expression has

embodied components due to the subjective approaches of the transcendental ego.[160] These approaches concern lived experiences where the becoming of objects in intuitive ways on behalf of the learner amounts to constituting *freely* within the broad mathematical field.

- An example of these expressions concerns the intuition of the general formula (§ 3.4.2.3.4) as a course of intuitions, realised after informal and formal intuitions, based on the former, regardless if previous instruction has taken place.
    - on *symmetry* (§3.4.2.3.1)
    - on the fallible intuition that the curve will not cross the x-axis,
    - on the x^2 *category* of curves (the *categorial* intuition), as based on the image of a bell, after 5 points of the curve were at place
    - in the formal treatment of numbers in the construction of the formula (but) with an *embodied* intuitive eye in keeping number 10 as central, as in using the point (10,0) in order to verify her initial hypothesis, linked to her embodied—distance from the wall—feature. ,
    - and finally, on the line(s) that the general formula depicts as *paths* next to walls, *comprised by points* in real phenomenological space (extract in §3.4.2.4).

  How all these informal intuitions or intuitional components are involved in the intuition of the general formula? Simply because they participated (*co-presentified* in Husserl's terms) in passive synthesis, as principal agents (vehicles) in shaping the horizons of the intentional lived experience that yielded the general formula, culminating in the formula with number 10, which become the *leading example* in her *synthesis of coincidence*. Since embodied consciousness is meant here beyond

[160] Transcendental, i.e. surviving as identical due to retentions and protentions that transcend recollections and anticipations alike, due to the conviction, the certainty of intuition that fills them selectively)

The following Table 2 summarises the main themes analysed in the data, and the last column shows that there is evidence that Mary has produced significant activity in all those areas, which was one of the main reasons that her work was chosen for this paper.

The task's main objectification features, according to the intuitive *acts* and *expressions* of these acts, as they surfaced for each intuitive object			**Mary**
Intuitions (Husserl's main categories)	**Intuitive expressions**	**Intuitive objects**	
EMPIRICAL INTUITIONS (Anschauung)	**informal logic**	▪ Symmetry of the curve.	✓
	kinesthetic	▪ Bird's-eye-view intuition. ▪ Points belonging to the curve are located by parallel lines to the wall (parallel lines intuition).	✓
	visual	▪ The diagram as the representation of the activity in the class (bird's-eye-view intuition). ▪ The x-axis is a boundary for the curve. ▪ The shape out of the 5 points (empirical stage of the x^2 intuition)	✓
ABSTRACT INTUITIONS (Categorial intuitions, and Wesensschau or *intuitions of essences*)	**visual**	▪ *Categorial intuitions*, in particular the second stage of the categorial intuitions—e.g. the shape that emerged from the 5 points (i.e. the first, empirical stage) seen as an x^2 curve (the abstract, categorial stage of the x^2 intuition.	
	mathematical–formal logic	▪ *Intuition of essence*. The constitution of a new, general formula, covering all real distances from the 'wall', in the three steps of the starting or leading example, the variation of the starting example and the synthesis of coincidence, due to the stereoscopic vision.	✓

Table 2. Summary of the findings in Mary's case

4.3.2 Discussion of Mary's data analysis key issues

The research supports the claim that phenomenology, being a philosophy of the 'immediate experience' can give a convincing account of the significance of intuition in the objectification process.

The two-level structure of intuitions (empirical/abstract), as it was introduced in section 5 of Part 1 (the theory chapter), and as it was analysed in the data (e.g. §§ 3.3.1, 3.3.2, 3.4.2.3.2 of Part 3 (the data analysis chapter) corresponds to ideas expressed in non-phenomenological literature. For example, Fischbein points at primary intuitions as the cognitive beliefs that develop independently of any systematic instruction as an effect of their personal experience, which corresponds to the Husserlian empirical intuition. And Husserl's abstract intuition is similar to Fischbein's secondary intuition (Fischbein, 1994, p. 68) that has no natural roots, and the Piagetian operational intuition (ibid., Ch. 4) that comes from special action towards objects of the world, rather than mere perception. By employing the Husserlian theory, the similarities and differences between the two levels of intuitions were clearly shown in the data analysis, without any appeal to idealist or biologistic concepts, such as the concept of the adaptation mechanism (ibid. p. 59), but by taking into account their position and functioning in the objectification process. The theory of intentionality that the Husserlian phenomenological perspective has afforded us—by distinguishing the *intentionality of act* from the *operative intentionality,* as well as their products and functions—allows us to sense the importance and the richness of the latter intentionality, appreciating its contact with the pre-objective raw (hyletic) material and its unfathomed synthetic capabilities.

But what is principally new in Husserl's approach is intuitions as *objectifying acts,* which separate new objects from unstructured, unthematised raw material:

- The bird's eye view intuition brought up a figure from the 5 points that Mary had found; this new object-like entity derived from unstructured sensory-motor classroom and home experience, and
- The general formula appeared as the identification between elements

> of particular formulas, which became essential only due to the fulfillment of the intentionality of the general formula. These formulas were grounded on an embodied perception of number 10, as "the distance between the fixed point and the wall". Moreover, the intuition of the general formula was realised as soon as number 10 was qualitatively transformed through its diverse conceptualisations by Mary.

Therefore, the phenomenological approach that I employ in my research deals with the learner's lived experiences as separations of lived profiles of the learner's reality, not as already categorised Kantian perceptions. It is now useful to summarise our overview of empirical and abstract intuitions, after the analysis of the findings:

Empirical intuitions are more than mediators of our lived-life, they are accumulations of experience of our embodied consciousness and they bring new objects that we *come to learn as we use* rather than construct. They are embodied, pre-objective operations, *followed* by action. Empirical intuitions concern the transformation of the sensory, unreflected, raw data to forms, to empirical objects, which become the ground, the subsoil of the abstract ones; (e.g. the 'bird's eye view' intuition that transformed Mary's classroom experience into the diagram and became the ground for her further abstract intuitions (§ 3.4.2.2 in Part 3).

Abstract intuitions bring objects of a new kind with them from 'ordinary' singularities, such as the figure that emerged from the 5 points on the piece of paper and was linked to the x^2 curves (§ 3.4.2.3.2 in Part 3), or the numbers—distances on her diagram—that were intuitively targeted and evolved to a technique and then to a variable, in Mary's general formula (§ 3.4.2.3.4 in Part 3). Abstract intuitions were shown:

- to follow similar patterns of emergence to corresponding empirical intuitions: e.g. the figure that emerged from the 5 points appeared on the ground of the grid lines of the paper and other lines she had drawn, in the same way as the x^2 curve graph emerged on the ground of other graphs (§ 3.4.2.3.2)
- to work on the products of empirical intuitions: e.g. Mary has recognised a

pattern in the 5 points, which became a basis for the x^2- curve abstract intuition (§ 3.4.2.3.2), or e.g. number 10, intuited as a critical number—as a key number for the progress of her investigation—and then becoming the basis for an abstract intuition, through its involvement in the formula production (§§ 3.4.2.3.3, 3.4.2.3.4), and

- to bring new sorts of objects to the surface (e.g. the general formula in §3.4.2.3.4), by focusing on invariable features of previously constituted objects (the two formulas), in the context of their cultural (mathematical) horizons.

Mary's last intuition towards the formal expression of the parabola (§ 3.4.2.3.4) is considered as a particular kind of abstract intuition, the *intuition of essences*. It illuminates the essential properties of two previously objectified forms; but the interaction between forms keeps the starting example as the principle knot of reference. In this case the general formula appeared as the essential property of two particular formulas and the first of them stood as a reference to Mary's embodied experience. Thus the synthesising intuitions of coincidence (essences) become the embodied, personal and at the same time cultural agents of learning and teaching experience. The intuition of essences revealed not just an abstract knowledge of formula, but rather a node in the system of layers of objectified experience grounded in the *living-body*.

I consider Mary's last intuition towards the formal expression of the parabola (section 3.4.2.3.4) as a particular kind of abstract intuition, the *intuition of essences*. It illuminates the essential properties of two previously objectified forms; but the interaction between forms keeps the starting example as the principle knot of reference. In this case the general formula appeared as the essential property of two particular formulas and the first of them stood as a reference to Mary's embodied experience. Thus the synthesising intuition of coincidence—and the essence it brought in the shape of the general formula—became the embodied, personal and at the same time cultural agent of learning and teaching experience. The intuition of essences revealed not just an abstract knowledge of formula, but

rather a node in the system of layers of objectified experience grounded in the *living-body*.

From one hand it is the *living body,* the body-subject that lies at the core of intuitions, from another hand it is the intuitions, the starting points for the discursive nature of knowledge, since they link the pre-objective raw material (hyle) of lived-experience to the reflective conscious treatment of cultural (mathematical) objects. During the objectification process objects emerge once the students become conscious of the effective implications of their intuitions and then the objects evolve through their further involvement in the students' activity towards the next objectifying acts, in order to arrive to the final cultural form. It is in this sense that intuitions of essences, as well as all abstract intuitions are the end points of the learner's analysis and "by the same token the starting-points of our epistemological and constitutive processes" (Hintikka, 2003, p. 180).

We can see here that one of the novelties of Husserl is that he gives a legitimate answer to Meno's paradox of the acquisition of knowledge, through the subjective cognitive praxis, which is transcending its cultural mediation at the same moment that it is culturally transcended by pre-existing tradition and similar constitutive practices. Being deliberately bracketed by phenomenological attitude cultural conditions took significant part in the investigated process of objectification: it was the cultural practice of modelling that allowed Mary to draw the diagram; it was the ruler as a cultural tool that allowed her to depict a straight line on the paper; it was the specially structured grid paper that facilitated the application of the coordinate system and her visual intuitions, and so forth. But it was only personal lived experience that let objectification happen, that made possible the coincidence of intuitive synthesis in Mary's last intuition. Thereby I suppose that Husserl, who has radically rethought Kantian and Cartesian dichotomies, could be considered as complementary to current cultural-historical approaches, from the point of view of the subject-in-the-world.

I exemplified how the lived experiences genetically break down the dualism of "outer" and "inner" world, since they are "appearances-of-something… of

existing things presenting themselves" (Held, in Welton, 2003, p. 28). *"[T]hey are the In-between, that which originally opens the dimension of intentional appearance within which consciousness and the world have already met—before any subject-object rift" (ibid.).* It is in the beginning, in the origins of objectification where the genetic issues of the constitution of temporality (Husserl, 1991, 2001) and of the "unity of coincidence corresponding to the unity of sense" (2001, p. 44; cf. 1970b, 1973) arise. There—in the origins of objectification—are structures that persist in personal cognitive experience, namely the synthesis of coincidence, the recognition of the generality from the mirroring of a starting example, and the double operation of categorial intuition. This double operation of intuition (§ 3.4.2.3.2) starts from an empirical particular that is constituted—not merely found yet not constructed—and then used for the categorial association in order to introduce the presented curve figure to the x^2 family of curves. Making association possible is a *poetic moment* (Radford, 2010, p. 6), which Husserl's theory of experience makes visible, where a student is engaged with her living body that manifests itself in its actions, starting with its intuitive body-subject-in-the-world constitutions.

4.4 General discussion

a. Summary and key issues of the data analysis

Now that the analysis and the discussion of each of the three learning episodes of Ivan, Diana and Mary have been exposed, a broader (general) discussion of the three cases as a whole may commence, in the context of the research questions that the study has posed from the beginning. It is a discussion that includes the following foci:

1) An account of where the main theoretical terms introduced in Part 1 were exemplified.
2) How the methodological routes introduced in Part 2 enabled the results.
3) The structures emerging from the close inspection of the inaugural stages of objectification in the findings themselves, in Part 3.

4) The introduction of Meno's Paradox, concerning the actualisation of new knowledge, which will receive a suggestion for an answer from the theoretical standpoint of this research, and
5) A response to contemporary and previous (yet popular as 'common sense' sort of understanding) constructivist/Kantian-based research frames.

The subsections that follow include discussion of these topics; they will illustrate the efficacy of the adopted theory and accentuate critical problems concerning other perspectives thar have already been indicated in the theory (Part 1), and evidence has been collected after the analysis of the learning episodes (Part 3). The argument developed throughout the research is based on the perception that the source and the ground of learning is the boundless possibilities that lived reality reserves for the learners, rather that its representations. It is the same problem when we deal with mathematical signs, as soon as we tend to forget that the sign is limited to a particular world that the object always transcends (e.g. 'three' transcends its significations III, 3, $|\mathcal{A}|, \text{when } \mathcal{A} = \{a, b, c\}$, a, b, c $\in \mathbb{Q}$). Learning in this sense is equivalent to separating associations of lived (previously intuited) perceptions of lived experience, and developing them to abstract mathematical objects that are meaningful for the task-at-hand.

In other words, it is presentations (based on immediate perceptions of what is present) interwoven with re-presentations and presentifications,[161] which are shaped as new thematisations and can be intuitively produced at any conscious moment, that make our cognitive ground 'groundless', full of *intuitive certainty* and the awareness of *indeterminacy,* driving us to new understandings through *closer determination* of already constituted objects and through new objects that we constitute and get to know as we use them.[162] The distinction of my Husserlian

[161] Re-presentation and presentification (see next footnote) refer to acts "whereby things are given in a less fully present way, as for instance, in acts of memory, imagination, symbolic representation, and so on" (Moran & Cohen, 2012, p. 260), in contradistinction to *presentation*, which is based on lived experiences "in which the intentional objects are given in intuitive experience, immediately, directly, and with *in propria persona*, 'in-the-flesh' *(leibhaftig)*, here-and-now, full presence" (ibid. – also c.f. Index-Glossary).

[162] In regular perception for instance, the object is presented through an adumbration (a profile of the real), but the absent sides which are co-intended are given through an empty presentification, in Husserlian terms.

phenomenological perspective from Kantian and constructivist views will be clarified, following the aforementioned argument line. And it is the only way according to the arguments employed here that the reactivation of mathematical objects can be observed as it operates; that is reactivation as the reverse process of the sedimentation of meaning, of sense.

4.4.1. An account of where the main theoretical terms introduced in Part 1 were exemplified.

Now that the analysis and the discussion of each of the three learning episodes of Ivan, Diana and Mary have been exposed, it can be realised that various terms of the theory found concrete description in the data, in various types of intuitions performing as critical objectifying acts. And the account of such terms and the instances of their descriptions in the data analysis will help in structuring a summary of the thesis, in relation to its analytical contribution. So the main theoretical terms that this study introduced and that have found detailed descriptions in more than one of the cases are the following:

- Intuition, in its various forms:
  - *Empirical* intuitions:
    - Diana's first stage of her categorial x^2 intuition.
    - Mary's bird's-eye-view intuition.
    - Mary's intuition of the symmetrical points,
    - Mary's intuition of the lines parallel to the wall.
    - Mary's first step of her x^2 intuition.
  - *Abstract* intuitions:
    - *Categorial* intuitions in two stages, namely the empirical and the abstract stages:
      - Diana's intuition of the rule, followed by the intuition of the algorithm.
      - The two stages of Mary's x^2 intuition.
    - *Intuitions of essences* in three steps
      - Ivan's intuition of the 'asymmetrical' diagonal pattern

- Mary's intuition of the general formula.

- The *interrelation between empirical and abstract intuitions,* in both cases, namely in the case of categorial intuitions—since the two stages that comprise categorial intuitions stand for empirical and abstract intuitions respectively—and in the case of the intuition of essence (compare Ivan's empirical focusing on the $3 \times 3, 4 \times 4$ grids' pattern—as a pattern worthy for exploration—and his abstract synthesis of coincidence at the third step of his intuition of essence Part 3, Ch. 2).
- *Objectification,* given through a literature example[163] and in various instances in the three analysed episodes, in the cases of empirical and abstract mathematical objects. In a nutshell, it was shown how objectification originates in the operations of the lived body, while different kinds of intuitions, driven by qualitatively different intentionalities, partake the whole process.
- The *living body* or *lived body* (used interchangeably), as the body-subject in lived-experience, particularly as Merleau-Ponty took it up from Husserl:
  - In Ivan's building up of his intuition of essence (during the first and second step of the intuition) and in his stereoscopic vision (that actualised the third step of the intuition).[164] Also in his intuitive last reflection in the classroom, when he felt that a solution is within his grasp (Part 3, §3.2.2.2).
  - In Diana's transformation of her empirical trial-and-improvement method to an algorithm. It was a case where the learner's intuitions actually record her thought process in the texture of the intuitive objects that she devised, which is a symptomatic feature of intuitive acts in general, pertaining to their operative and active involvements in objectification.

[163] Cf. the 'red book example in Part 1, § 1.3.3.
[164] Cf. Part 3, §3.2.3.

- In Mary's transformation of the diagram to a classroom bird's-eye-view and various other cases up to the end, when the general formula arose from her 'vertical' link with 10 as "the distance from the wall", and as she was describing the property of the general formula by linking it with her embodied experience, as a line where "you could stand on anywhere on the *line…* and you would be equidistant from the wall and a fixed point".

- *Operative* intentionality and *intentionality of act,* in cases such as:
  - The categorial (abstract) intuitions of Mary and Diana, where the aforementioned intentionalities find expression in the two stages of this abstract kind of intuition respectively (*operative* in the first, empirical stage and *intentionality of act* in the second, abstract stage), as Husserl's theory anticipated.
  - Ivan's intuition of essence, where *operative* intentionality was detected running in all three steps, from the persistence to the 'asymmetrical' pattern, up to the third step of his intuition, which was actualised by the *stereoscopic vision* of the two first steps, in the so called *synthesis of coincidence.*
  - Many 'small' examples, like
    - Ivan's shifting of intentions from symmetrical to asymmetrical patterns already during his classroom exploration, after he found the asymmetrical 4×4 grid.
    - Diana's work out of the task's rule as a set of algorithmic commands that resembled her trial-and-improvement process, as she separated the *form of the algorithm* from the 15 drawings.
    - The ways that her intuitions of symmetry and of the x-axis as a barrier for the curve interfered with Mary's investigation.
- The process of closer *determination* of the objects that the intuitions brought to the surface, as well as the objects themselves shaped the particular approaches that the students adopted in each of the three learning episodes:

- In Ivan's case, when his determination to follow symmetrical patterns throughout the classroom investigation hindered him from realising the significance of a pattern that he had found, and how his intuition of the asymmetrical diagonal pattern at home changed everything.
- In Diana's case, where her intuition of the task's rule inaugurated her algorithmic approach, followed by her intuition of the algorithm that depicted the steps she went through, up to her flowchart.
- In Mary's case, when her bird's-eye-view intuition set the ground for her subsequent embodied treatment of the task or, more precisely, for linking (communicating) her (fresh in memory) classroom experience with her intuitive mathematical experience, through her "little diagram". As Aristotle has put it, "The greatest thing by far is to have a command of metaphor. This alone cannot be imparted by another; … for to make good metaphors implies an eye for resemblances" (Aristotle, *Poetics,* 1459.a4).[165] And after the analysis and discussion of Mary's learning episode as a whole, we have seen where Mary's command of her bird's-eye-view intuitive metaphor has led her. She indeed took command of her bird's-eye-view intuitive metaphor. Full of intuitive ideas and rich in producing tools that brought more intuitions and objects respectively, she culminated her investigation with an essential objectifying act that brought an object of a new *category* to the surface (categorial act), by revealing an *essence* of her first formula, in its generalised version. Thus *generality* and *apodicticity* (quality pertaining to proof) were detected as crucial features of the Husserlian intuition of essence, for the second time after Ivan. The *stereoscopic vision*—which is now posed as an immanent feature of the intuition of essences—was confirmed once

[165] I find exciting that this fitting Aristotle's extract to Mary's intuitive grasping of the bird's-eye-view metaphor comes from his *Poetics*, due to Radford's 'red book' example, which was analysed in Part 1,, §1.3.3 as a prelude to the analysis of objectification, and as an exemplification of the transcendental phenomenological reduction that is also adopted through the research.

> again as she mirrored the whole process that led her to the first formula, into finding a second formula, and as soon as it was done the operations of the lived body synthesised a variable to replace 10 primarily and 6 secondarily. And it was also shown for the second time[166] how the intuitions of essences are in fact categorial intuitions, thus putting (successfully) the theory to the test of actual learning praxis, since theory contends (cf. Lohmar, 2010, p. 117) that the intuition of essence is a form of categorial intuition (c.f. Part I, Ch.).

From the accounts of the lived body and of the intentionalities analysed in the data and the abstract intuitions (categorial and intuition of essence) that the *operative* intentionality precedes the *intentionality of act* in every instance, as expected from the theory, with the former intentionality being an expression of the synthetic, associative and orchestrating performance of the *lived body,* while the latter intentionality being associated with the objectifying, constitutive stage that cognition is recognisable as such, entering the realm of communicable expressions. Ivan, Diana and Mary brought evidence (in the phenomenological sense) by clarifying different adumbrations of different expressions of the same phenomena, namely intuitions (empirical, categorial and intuitions of essences) and their corresponding empirical and abstract mathematical objects. Thus we investigated closely how intuitions in each case operated as *objectifying acts,* endowing with meaning (sense) the learners' lived experiences.

4.4.2. How the methodological routes introduced in Part 2 enabled the results

In the analysis of the data the aim was to investigate the genetic formations of sense (Husserl, 2001) and discover the presuppositions of logic and formal mathematics in the learners' lived worlds. For this reason the phenomenological methodology was employed, as the one afforded with the critical questioning of the world's appearance as 'already there', while this very world is inevitably constituted

[166] Since in Mary's case the 'essence' of 10—from the distance from the wall to a constant in a formula and then to becoming a variable—revealed the object of a new category, i.e. the general formula.

as such, as 'already there'. By not reducing the learners' world to a *phenomenon,* in an acknowledged, originally Husserlian sense, we would be 'blind' to its constituted nature and unable to discern the structurations that resist psychologistic and naturalistic exigencies.

Already prepared from the theoretical part, we were warned of the pitfalls of (Kantian informed) structuralism, which inevitably resorts to an unfounded intrumentalism, deemed to achieve local results, in borrowing material from other methodologies (e.g. RME, Activity Theory, enactivism through Varela, Rosch and Johnson); thus acquiring a vague link to phenomenological and materialist dialectic roots. On the contrary, and distinctively to contemporary research that employs phenomenological conceptual frames (Roth, DeFreitas & Sinclair) the starting point here is that "perceptual experiences are intentional because they are a revealing or disclosing of the world. And they are conscious because they are a revealing or disclosing of the world to someone—their subjects" (Rowlands, 2010, p. 163). And from the fact that intentionality, as revealing the world and disclosing it at the same time with indeterminacy, already starts with awareness in a double sense: that is the awareness of the aforementioned indeterminacy and the awareness of the mine-ness of the perceptual experience:

> If I am aware of a conscious state and/or its properties—I am aware of my pain or what it is like to have it—then my conscious experience at that time consists not just in that of which I am aware; it consists also in my awareness of it. (ibid.)

So, as now can be clarified, the methodological routes described in Part 2 and depicted on the Graph 4 in § 2.2 were informing the theory as they were enabling the data sources to become data, the data to become findings and to become a new sort of data for further reflections and further findings. This general structure is only a rough description of processes that shifted from key ideas to intuitions, then to the operative intentionalities and the intentionalities of act that sustained these intuitions, then to the lived experience that was the source for the embodied consciousness that availed these intuitive operations.

The regressive course that the reduction followed was from the mathematical objects to the powers of the lived body: from Ivan's nine formulas to the 'asymmetrical' diagonal pattern, to the quest for symmetry, for a unifying pattern that 'ought' to appear on square grids, throughout his classroom investigation. We can trace here the successive reductions leading to the intentional origins of Ivan's nine formulas: from the formal mathematical objectivity of the formulas, to Ivan's lived experience of symmetry and square grids, there is a strenuous path that Husserlian phenomenology undertakes step by step, since it recognises passivity (the pre-reflective and pre-objective stage) as the field for the intentional forces of the lived body. And since it deals with the transition from empirical to abstract knowledge without the separatist Kantian/Piagetian frame of sensibility and knowledge.

And from the 'other side', namely from the viewpoint of the research's methodological apparatus I started with the observation of Ivan's investigation of the task in his group, and his and Carry's short interviews, taken in the middle of the session; the contrasting results that he produced and the "breakthrough moment", as recorded in his coursework, rang a bell that this data source could become data. After the analysis of a lengthy interview with Ivan I disclosed the intentional origins of his mathematical objects. And it was only then that the reactivation of Ivan's course towards his decisive intuition became available for phenomenological description. After confirmation e-mails and successive reflections the structure of the intuition of essence surfaced, as the photographic image appears slowly on celluloid, in the darkroom, and it was later confirmed as such by the theory. Successive analyses, where findings were becoming data for new findings brought new elements to the surface, as the three features/properties of intuition [167] were applied in the three-step process, up to the synthesis of coincidence. It was from this elemental unit of phenomenological knowledge production (see Graph 4 in Part 2, § 2.2), as the phenomenological attitude was

[167] Namely immediacy, intentionality and the feeling (filling) of certainty.

locked on to the phenomenon[168] of the emergence of objectification, that notions such as the *stereoscopic vision* appeared, in order to maintain the sense of Ivan's intuitive object (the 'asymmetrical' diagonal pattern), before it entered the mathematical domain—thanks to the arithmetic rhythm 1, 1, 2, 1, 1, 2, 1, 1, 2.

A pending question run through the study, concerning mathematical objects born out from tension, as this fundamental awareness of determinable indeterminacy was cashing in intentionalities that operated pre-reflectively, before they were reflectively constituting. Such was the concern of the phenomenological analysis, to trace these intentionalities from their neglected depths, as they shaped and availed the objects that finally appeared; aiming at disclosing the genetic moments of objectivation, as the primal stage of objectification. This absolute focus invites reduction and bracketing, whatever the theme of phenomenological research may be, in micro or macro level, if a meaningful description is to be presented, with all the care that the bracketing will be neither arbitrary nor excluding (what is bracketed). Particularly in the case of my study this rule was followed in detail, since the teacher was naturally abstained by any form of guidance towards the students, and I followed his example, choosing cases to analyse that interference from both me and the teacher was truly minimum without being forced, even in a remote sense. Therefore the bracketing of teacher and researcher came as a given principle of the course, adopted by the researcher in a naïve exploratory sense, as I did not know what it was that I was looking for when I was participating these sessions, before the transcendental phenomenological reduction and bracketing were deepened and explicitly implemented, as a route, as a becoming, rather as a state of affairs.

[168] That is phenomenon in the phenomenological sense that this study has introduced; and not in the commonsense use of the word, which is often confused with its specific employment by the phenomenological study of intentional consciousness. It is common practice in our field to use phenomenological language without any justification, as if phenomenological terms have not been historically introduced (e.g. Coitchu, ESM, 2015). Such an indifference towards historically introduced terms would most probably be addressed accurately by phenomenalism rather than phenomenology.

Finally, the extensive use that the structures of categorial and intuitions of essence may afford to mathematics teachers, for their abstracting and generalising capabilities, in teaching and introducing new mathematical concepts, as well as new approaches to old concepts by the students,[169] in a critical analysis that involves their lived worlds.

Now that the methodology and the methods are opened up for further reflection we can say that the *initial reduction* was put into action as a result of the decision to follow the Husserlian inversion of the Cartesian way (Part 1, §§1.1.b, 1.1.d, 1.4.2, 1.4.6; Part 2, §2.8), consisted of the beginning with the being of the objects of thought (the *cogitationes*), of experience as it occurs and is simply being reflected upon, and of taking the objects of thought as the ultimate source that cannot be doubted (Husserl, 1999b, p. 62). This reduction of the cognitive praxis to a phenomenon in which its primordial act is already a (proto)cognitive act, accordingly entailed bracketing (of the teacher and of the participant observer), in order to render the students' objects of thought visible, to 'isolate' them so to speak as components of the primal relation that the fulfilment of intentionality renders possible.

Thus the research question concerning the possibility of analysing mathematical objects as phenomena of lived reality, and not as ready-made objects of our world, 'already existing' and 'ready to be transferred', was answered by the phenomenological theory employed in the study (see Part 5, B), but only after the phenomenological methodology was put into action, with reductions and bracketing as aforementioned. In philosophical terms, the phenomenological attitude and the concomitant reductions and bracketing allowed me, as a mathematics education researcher, to think differently about the issues of learning mathematics, in particular of acquiring abstract knowledge through tasks; which was the reward for

[169] In a process reminding me of Ambrose Bierce's 'old saws fitted with new teeth', where Aesop's myths receive new take-ups.

trusting the transcendental (intentional) ego and the constitutive ground of the lived body and its powers.

In small coins, the absolute focus on the intentional web sustaining each student's objectification, allowed the tangibility of the inaugural intentional reflective modalisation that turned perceptions to judgments, namely constitutive judgments.

Was not Ivan's posing of the 'asymmetrical' pattern as the pattern for all grids such a constitutive moment? Diana's posing of the algorithm, as an essence of the trial-and-improvement component of the rule of the task? Or Mary's posing of the "little diagram" as an embodied bird's-eye-view over the classroom, allowing her a different look over the maze she was into literally (when she was enacting the fixed point in the classroom) and metaphorically, according to Meno's aporia, as she knew nothing about what she was looking for. Nothing, apart from the fact that it should look like a curve rather, a straight line! Which, although erroneous, it was already indicating an object. All this marvel of embodied experience, in each case of the analysed episodes would simply not be visible without the reduction and bracketing that turned our regard exclusively on the intrapsychic processes and their links to the lived (embodied) reality that enabled the objectifications.

The phenomenological approach of the research suggests that unfolding the multiple stratifications of the subjective perspective of the learner's lived-world is equivalent to letting actual engagement of the learner emerge, and the learning praxis to take place and be recognised as such. Focusing on the students' intuitions and objectification processes is an approach that leads to the core of their learning practices, since intuitions, as particular expressions of perception, are connecting the hyletic data, i.e. the "unstructured raw material"—that is 'material' with previous or just formed significations, and in any case taken up anew—with "our structuring and form-giving mental activities" (Hintikka, 1995, p. 98). Nurturing the students' intuitive processes, facilitating them with corresponding intentional teaching adjustments, and aiming for the "intuitive starting point" (van Hiele, p. 122) are suggested forms of action.

The backbone of such practice as the one I just suggested is no other than the methodology/methods that need to be employed, as I also stated earlier. The methods of the *phenomenological reduction* and the *bracketing* (*epochè*), and the methodology of the *phenomenological attitude,* as a process of becoming (a "zig-zag" movement)[170] from the natural attitude, "our straightforward involvement with things and the world", to the phenomenological attitude, "the reflective point of view from which we carry out philosophical analysis of the intentions exercised in the natural attitude and the objective correlates of these intentions" (Audi, 1999, p. 405). These methods allowed the "'leading back' from natural beliefs to the reflective consideration of intentions and their objects" (ibid.). And, most importantly, they allowed access to the part that intuitions play in this fundamental cognitive unit, which is cancelled (as a Cartesian dualism) as soon as the lived body and its powers[171] become a recognised principal actor. It is after pondering over the manifestations of the lived body[172] that one is legitimised to substantially refer to cognitive actions (as a fully-fledged praxis) or constructions in general, without risking becoming groundlessly functionalist.[173]

Let me put it this way: how far would the detailed analysis of Ivan's mathematical activity go in revealing the sense of his mathematical activity, not only in the significance of his new object for the task, but also for us, as researchers trying to grasp a clearer sense of the cognitive praxis? Would there be a clear distinction of the prevalence of the lived body in re-cognising the 3×3, 4×4 grid pattern (before finding the arithmetic pattern), if the *intentional origins* of Ivan's intuition would not be explored as the origins of the sense endowing forces, if the intentionalities (operative and active) that made his intuition a "breakthrough

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[170] Husserl, 1936, pp. 50, 177; 1970a, p. 58; 2001, p. xix.

[171] The lived body and its powers serve the embodied consciousness as an organ of perception, substantiating our metaphors and shaping our thoughts.

[172] A detailed approach of the manifestations of the lived body in students' diagrams is employed in the work of DeFreitas & Sinclair (2012) Although comparing and contrasting my perspective with DeFreitas' neo-empiricist approach (e.g. add refs) is beyond the scope of this study, it may suffice to say that her detailed research in diagrams, inspired by Châtelet's radical phenomenological viewpoint, is perceived as compatible to the manifestations of the lived body that are mentioned here.

[173] Cf. following section §4.4.5.

moment" would not be prioritised and investigated—before we got to know that the 'asymmetrical' diagonal pattern was the fulfilment of his primary intention to find a pattern for *all* grids? Finally, would the detailed analysis of Ivan's mathematical activity reveal the *sense* of his 'asymmetrical' diagonal pattern, firstly as a pattern, recognised by his living body as suitable for all grids, and only then as exploitable mathematically through the rhythm (1, 1, 2, 1, 1, 2) of the vertical street cops? Since the sense is related to the operations of Ivan's *embodied* consciousness, namely his lived body. And it is also the *research essence* for that matter, of my persisting research questions, related as they are to the commencing of objectification.

The analysis of Ivan's intuition would make no sense from my phenomenological perspective if the analytical focus would not turn its regard towards the primal significance of the lived body in objectification. By recognising and prioritising it we managed to realise how Ivan's lived body came first, and played a crucial part (firstly) in pursuing a symmetrical pattern for all grids exclusively in square grids, and then again, in perceiving *stereoscopically* the *synthesis of coincidence* of the square and the $3 \times n$ grids.

Such self-fulfilling rhetorical questions emanate from each of the three cases. Ivan's case is only the first example where my particular attention to the lived body and its powers gave results. Another case is Mary's birds-eye-view intuition, which presupposed and consequently manifested in various ways its embodied (kinesthetic and visual) character; it was another case where operative intentionality was proved crucial, since it allowed various re-presentations of the classroom setting, hence facilitating subsequent intuitions.

The friendly and open exercise of the research experiment that I was engaged in for the last four years was an experiment that took place for the first time, for all of us, the researcher and author of this dissertation, the teacher and the students, without any prescriptions or full control or imposing of any kind of attitude, apart from the instruction not to interfere in the students' strategies or their be-puzzled moments. No methodology was pre-given, but the phenomenological methodology

became inevitable as my initial aim was to get as deep as possible to the genetic 'moment' of *objectification*. That is to the moment when objects appear anew for the learners, as they interact when working in groups, allowing me to observe and record their mathematical investigations.

4.4.3 Structures emerging from the close inspection of the inaugural stages of objectification.

We followed Ivan's investigation up to his constitution of the 'asymmetrical' diagonal pattern as the appropriate pattern for all grids, after the exchange of ideas in his group and his focusing on symmetrical patterns. We saw how he found the pattern but did not understand its significance. And then, we analysed the rich three step process of the intuition of essence, which comes about due to the synthesis of coincidence that is achieved in the third step. We investigated the transition from the raw material of the pattern as it appeared on paper, to the abstraction of the generalised "shape pattern". Since we revealed Ivan's initial intention for a general pattern for all grids, to which this object was constituted for.

In Diana's case we saw the desperate attempt to constitute something out of her own understanding, which was focused on the "rule" of the task itself. Without moving away from the trial-and-improvement method that she was extensively using throughout the sessions she devised a set of commands, which became an algorithm, and finally a flow chart, depicting the rule as it was applied for *any* number. Diana effectively used a categorial intuition in two stages, the empirical and the abstract, which yielded, due to a double move of her intentionality, firstly the rule and then the algorithm (cf. Part 3, §§3.3.1, 3.3.2), without anyone's help in the class, before she elaborated her idea with a flow chart at home. The teacher legitimised her answer instead of rejecting the idea as off-purpose, since the aim was supposed to be the exploration of different numbers and according generalisations, and the revealing of the properties as a result of this exploration (see Figures 11 and 15 in Part 3, §3.3.1, for three different approaches). There is no doubt that the teacher's legitimisation of her answer was the key to her major understanding after this activity, that "*because I've got a different answer to*

somebody else it doesn't necessarily mean that it's wrong, it just means I thought about it in a different way" (see Diana's interview in Appendix C). Intuitive objects that find their way to the formal mathematical domain bear their own truths, and hold their convictions, especially if they are encouraged by the teacher. It is in this sense that my view on intuitions is from the learner's perspective as it can be perceived by the teacher, as I am trying to understand the manners that lead to it, based on necessity rather than causality.

Cultural, social, political, ideological in any sense *horizons*, were bracketed—neither arbitrarily nor in an exclusive way—in order to allow the phenomenon of *intuitions as objectifying acts* to come to the fore. Although not under the scope of the current research, they may well become the theme of investigation in research with different scope than this one here. In other words, the argument is that phenomenological theoretical and methodological frames such as the ones adopted by this research may well be included in the arsenal of research with a different scope or even theoretical standpoint, as long as thinking acts and thought objects are under investigation.

In Mary's case we saw how an embodied start-up can help students bring up fluent embodied and intuitive activity. The history of Mary's concept of the x^2 curve appeared as full of small discoveries and misleading assumptions that were overcame by her subsequent intuitive acts.[174] In this sense, the individual learning history may be more instructive than the history of a concept. My research supports the teacher's desire to learn from the students' particular ways of learning, to communicate these individual perceptual approaches with the other students, and to reflect on her own. I offer my analysis of the students' unique learning experiences as starting examples for teachers who would venture to bring some implication of phenomenological attitude into teaching practice.

[174] She 'saw' that the sought curve is not a straight line from her bird's-eye view of the diagram (Part 3, §3.4.2.2.) and she removed her previously intuited x-axis barrier by applying her just objectified 'technique' of the parallel lines (§3.4.2.3.1).

4.4.4 The introduction of Meno's Paradox, concerning the actualisation of new knowledge, which will receive a suggestion for an answer from the theoretical standpoint of this research

The phenomenological perspective advocated by this study opens a possibility to disclose the mystery of "[h]ow will you set about looking for that thing, the nature of which is totally unknown to you?" (Merleau-Ponty, 2002, p. 431): at the very beginning of a new knowledge it is the living body and the lived world that meet beyond the object-subject dichotomy, in order to shape—more accurately to prefigure—objects by intuitive acts. It is the origin of the objectification process, when intuitions bring objects on the surface of consciousness, shaped exactly in accordance with the operative intentionalities of the learner, as it was shown in each of the three cases of this study.

The study of intuitions reveals the latent object-like formations of sensory-motor behaviour that emerge in the constitutive, reflective moment of objectification. As it was shown in Ivan's case, the symmetry 'syndrome' that occupied almost all of Ivan's experience in the class was shaped primarily by his lived body: we can induce it not only by what happened, namely by the decisive intuition of the 'asymmetrical' pattern, but even from what was not conceptualised until then, since he was judging each grid's pattern according to its symmetry; thus, the 5×5 grid that employed the 'asymmetrical' pattern of the 3×3 grid was (justifiably) classified as symmetrical but it was not classified as repeating the 'asymmetrical' pattern of the 3×3 grid, simply because his lived body did not recognise it as such. In other words his operative intentionality had a different direction than synthesising an asymmetrical pattern with a symmetrical one.[175] The asymmetrical 4×4 grid and the diagonal pattern that he later found somewhat reoriented his lived body, although not able to understand "at that point the significance of this". And it was quite different when his lived body focused on the $3 \times 3, 4 \times 4$ grid pattern at home, when he expanded from square to orthogonal

[175] In other words re-cognising their common pattern, beyond the dichotomy according to symmetry.

grids and employed his previously acquired technology in order to detect the 1, 1, 2, 1, 1, 2 rhythm in the vertical street cops. The powers of the lived body are clearly manifested by the contrast of Ivan's passivity—in the context of the 'asymmetrical' 3×3 grid pattern (before he found the 'asymmetrical' 4×4 grid pattern), as he was passively and actively directed otherwise—and his passivity and activity that followed, towards the 'asymmetrical' pattern. And it is an indication of how small examples[176] can work in revealing operative intentionality as an expression of the lived body, since "[w]e do not see the operation, since it operates. We see what would be missing without it; we circumscribe it as what makes speech be a 'speaking of'" (2002b, p. 44).

From the viewpoint of the mathematical tasks and the cognisant's actions and origins of actions that this study argues for, what is separated from the learner's lived reality is open in being re-cognised as the object that fulfills the intentions that will tether it to the task-at-hand, until the new object will acquire its position in the communicable frame of mathematics. Moves that are firstly realised in an embodied manner (e.g. gestures), are later transformed to signs (Arzarello, Ferrara, Robutti, Paola, 2005; Radford, 2003, 2009). But what is crucial for Meno's Paradox is that perception allows previously unperceived hyletic (raw) data to be determined in a new sense. And new knowledge to appear as previously unnoticed, intuitive gathering of sensuous and/or abstract configurations of the lived, embodied real, thus becoming available for communication with others—subjectivity turning towards intersubjectivity right from the inaugural stage of constitution, as its telos. Concerning our quest for an answer to Meno's Paradox I will use Husserl's words as a reflective instigator…

> In any experience, even that of an objectivity apperceived for the first time, a preknowledge of as yet unapperceived properties of the object is involved,

[176] Some of these small examples are mentioned in the account of *operative* intentionality and *intentionality of act*, in the earlier section §4.4.1. My study is abundant of such 'small examples' due to the employed methods and theory.

> a preknowledge which might be undetermined or incompletely determined as to its content, but which will never be entirely empty. (1970a, p. 94)

It makes us reminiscent of the ancient Athenians' temple to the unknown god, from which Paul preached in order to convert them to Christianity, claiming that he brought them the unknown god that fulfills their anticipation. Under the dialectic principle that action provokes reaction 'Saint Pauls' emerge in every unknown slot (or podium, if you prefer) that we have afforded our conscious acquisitions with. The 'Saint-Pauls' are compared to what is intended for the tasks-at-hand; moreover, podiums are endlessly constituted as soon as particular intentions arise, aiming at them. The knowledge/ignorance dilemma that Meno's Paradox puts us against becomes a false dilemma from my Husserlian viewpoint, since there is no such thing as sheer ignorance or complete knowledge in the first place: already at posing the question there is a prefiguring anticipation (protention) that already prescribes a solution, there is always "a preknowledge of as yet unapperceived properties of the object" (ibid.), and even after it is discovered the object is open to new thematisations, in a process of *closer determination* that can be taken up again and again. In the pre-reflective area where the object is prepared takes place an embodied (pre-reflective yet conscious) "pregnancy", where the hyletic data, the "real but not intentional element of the experienced" (Derrida, 1978, p. 205)" feeds in, open to unexpected thematisations that operate in intuitive certainty. It is not due to oblivion that is withdrawn that Meno's Paradox is resolved, as the Platonists would have it, nor due to biologistic justifications, in which Piaget-Fischbein's constructivist approaches resort. And the indeterminacy concerning the object of new perception that actually never ceases, is not a problem but rather a challenge, an intuitive drive in some sense for new objects and objectivities.

4.4.5 Discussion of approaches to perception in contemporary mathematics education research

The critical part that intuitions play in the objectification process is not an explored (thematised) topic in mathematics education research, despite the rich literature that draws on each of the two subjects, namely intuitions and objectification, for

their own sake. For it is one thing to talk about how important intuitions are and emphasise the contribution of intuitions to the construction of mathematical knowledge production, to pose an essential link of complementarity between intuitions and formal mathematical knowledge;[177] it is one thing to call for the awareness of gaps between "treasured intuitions" and actual mathematical truths (Fischbein, 1994; Kidron, 2011; Kidron & Dreyfus, 2010), and trying to bridge them (Ejersbo, Lenon &Arcavi, 2014). Finally, it is quite another thing to ask for better educational practice after experimental observations and pre-conceptions of the separate functioning of *sensibility* and *understanding,* without the crucial instrument of *intentionality* and the exploration of the powers of the *living body,* operating as an *organ of perception.*

And it is quite different to perceive objectification as an activity grounded on the pre-reflective, operative, silenced stage that Husserl calls passive stage, where the operations of the lived body prepare what appears, and protend what is coming; by protend here meaning a latent anticipation which may be fulfilled or collapse, and which is the foundation of new knowledge.

For example, it is quite different to detect the embodied roots of Mary's bird's-eye-view intuition of the diagram, who took command of her metaphor in realising the diagram as depicting the classroom setting that she had experienced a few hours ago. The release of a new perspective that the bird's-eye-view intuition brought her ensued the release in handling the task with formal, mathematical tools. It enabled the transition from her classroom experience and the real wall ('limited' in one sense yet 'rich' in another sense), to the mathematical exploration, without leaving her embodied experience in the classroom behind. This transition was depicted vividly in her interview:

> Once I could draw a diagram and look back [to the classroom experience], I look at it from *this* point of view; it's much easier than being in *that* point [i.e. the fixed point that she enacted in the classroom], that looking, like,

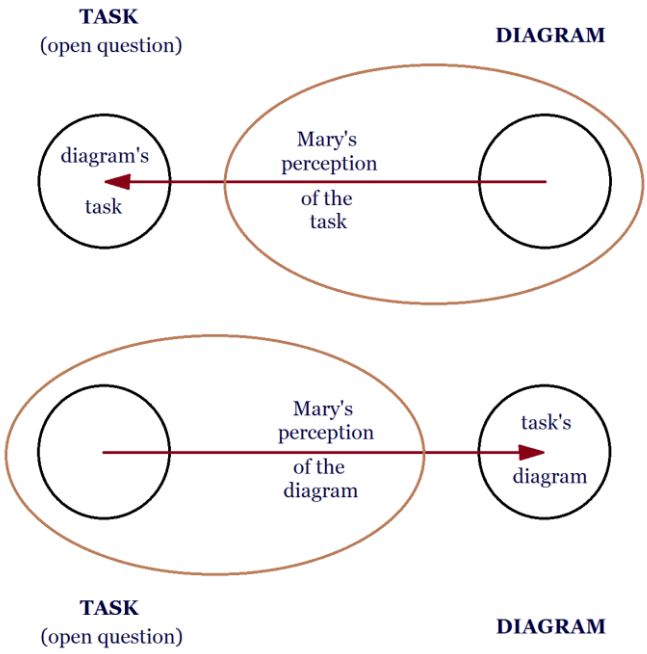
[177] *Mathematical* is another name for what has been called *abstract* throughout this study, which is concerned with the transition from the pre-mathematical to the mathematical knowledge constitution.

when standing here [the fixed point] *all you see is the wall!* You don't see all this is going on as well.

So what is the object here? What did this intuition of the diagram as the classroom setting brought to Mary's exploration? It's is not hard to perceive that an embodied operation is implied by the bird's-eye-view, set by kinesthetic, synthesising perceptions of the piece of paper, (Figure 17 in Part 3, §3.4.2.1), when only a straight line (the wall) and a point (the fixed point) were drawn. It is not hard to feel that this embodied operation was repeated again and again in her investigation, that it became a landmark, or rather, a source of further intuitions (like the parallel lines to the wall, or the startup of the x^2intuition).

But what was the object that the bird's-eye-view intuition brought to life, as this theory promises that intuitions actually do bring objects with them? Is it the diagram itself, seen through 'new eyes', after the ever renewed operations of the lived body? My answer is both affirmative and negative; affirmative due to the dynamic transformation of the diagram as new lines and points were drawn, as new horizons, in Husserl's terms, entered her field of exploration through the diagram. On the other hand the object of the bird's-eye-view intuition cannot be 'just' the diagram, since the diagram is only a *profile* of Mary's perception of the task, including the diagram. The task was a pole that determined certain choices to be made and others to be abandoned. But Mary's particular (subjective) response to the task's open question ("what kind of curve is formed?"), which is the primary focus here, not only motivated her transformed view of the diagram, but it also motivated her perception of the task (as finding new points that belong to the curve, until the formula could be extracted). This dynamic move (the shifting of Mary's horizons within her investigation) from task to diagram and from diagram to task was clearly manifested by Mary's extension of the task. And it is schematized in the following Graph 5, since it was a primary source of tension that fueled her

subsequent investigations.[178] Since it shows how Mary could shift from the 'I can' of the diagram, which enabled her particular approach of the task,[179] to the diagram as an object of Mary's perception in furthering the task's open question. In other words, the diagram did become a new object as it was transformed by Mary's bird's-eye-view intuition and at the same time the intuited object fueled a constant dynamic operation (through her lived-body) that transcended (extended) the diagram's objective sense, towards furthering her perception of the task's open question. And in the synoptic Husserlian vocabulary, the living body kept operating through the diagram (the transformation of the diagram was ongoing), either by entering the diagram's intuitive resources (living the task through the diagram) in order to achieve answers to the task's question, or by reflecting on the task's question itself, manipulating the diagram as an object.



Graph 5. The tension between the diagram and the task. The diagram becoming subject and object of Mary's perception, as it produces results for the task's open question and as it extends the task.

[178] This dynamic move also delineates the shifting of Mary's *intentional forces* (operative and active) towards two poles of her investigation, namely the pole of the open question that the task posed and the embodied possibilities that the diagram allowed and (to some extend) prescribed.
[179] The task becoming the diagram's task.

Mary's *bottom-up dialectic* towards the task gave results beyond the expectations of the task's initial question, since Mary's abstract, mathematical achievements remained grounded on her perception of the diagram. It is manifested in her interview, as she describes the property of the general formula that she found, as "standing on it"; which was a clear indicator that her perception of the curve was still grounded on her diagram, in other words this perception had remained the texture of her investigation. And it is in this sense that the object of her bird's-eye-view intuition transcended her perception of the diagram.

The aforementioned example was picked up from the variety of examples that this study offers, for a few principal reasons that are relevant to the arguments developed through the study, in order to exemplify my response to contrasting or seemingly similar approaches of intuition and objectification in our field, based on Cartesian and Kantian models of perception. The main reasons are the following:

The potential of the study of empirical intuitions, as placed within the context of intuitive objectification, and in particular the potential of the exploration of the constitutive aptitude of the lived body, which are themes that are largely ignored in our field and fully misunderstood by Kantian inspired research.

- The distinction between signification (e.g. the lines of Mary's diagram endowed with formal mathematical sense) and objectification (e.g. the 'I can' see the classroom through the diagram, from such and such perspective), which is the *source* of signification, echoing the prehensions[180] of the lived body that nevertheless transcends (traverses) all significations.

It is clear that the latter of the two reasons—based on reflections grounded on the data presented here—indicates and to some extend determines the scope of the former. In other words, the object as what transcends any of its mathematical significations (e.g. "three" as more than its significations III, 3, or $|\mathcal{A}|$ when $\mathcal{A} =$

[180] Prehension is "an imperfect or incomplete 'grasp' of a purportedly objective state of affairs, where it is somehow known that the state of affairs is imperfectly or incompletely given … where the content of the grasp adumbrates or points to something beyond what is given" (Tragesser, 1977, pp. 17-18, also cf. Part 1, §§ 1.4.3.1, 1.4.5, Part 3, §3.3.1, 3.4.2.3.1).

$\{a, b, c\}, a, b, c \in \mathbb{Q}$), yielding different profiles in different tasks $\left(3x - 2 = 0, \lim_{x \to 3} f(x)\right)$ effects my way of looking at objects, in particular abstract (mathematical) objects, as I live through them in mathematical activities. This is happening to me, to my students, to these three students and it is very much a common experience. Yet it is taken for granted and not questioned critically, as appearance synchronous to the appearance of reason; which is constituted, according to my Husserlian view. The question underlying my research questions is firstly, *how abstract knowledge becomes possible* in the first place; and secondly, *how close one may get to the origins, the genetic moments of abstraction*—more precisely *the genetic moments of abstract, mathematical objectivity*—through objectification.

My study brought theoretical and analytical evidence of an approach to objectification as a hand-in-hand operation with intuitions, in their interrelated empirical and abstract expressions, an approach that is theoretically grounded on Husserlian ideas. The critical aspect of the study starts from its perception of the intentional ego (whose conscious acts are always conscious *of)* as a starting point of the cognitive praxis, and from bracketing (without excluding) anything but this fundamental relation between what is *intending* and what is *intended*. That is reducing the experience to the fundamental relation between the subjective and the objective poles respectively—and most importantly, realising the interplay of these poles on the ground of *lived experience*. Thus the study focused on the organ of perception called the *living body* or the *lived body,* and its powers—like *operative intentionality,* which is responsible for syntheses that allow *objectification* to commence. This full view of the cognitive phenomenon, which the phenomenological theory and methodology made possible has now shown its potential towards Kantian/constructivist/Fischbeinian perceptions of objectification and intuition, through the analytical depth achieved and the explication of the transition from empirical to abstract knowledge, which are taken as separated though complementary by the aforementioned theories..

This research study is—considering its size and aims—an attempt to open up

an exploration of the history of abstraction, based on Husserlian phenomenological concepts, such as the ones introduced in the theory and met in the data analysis. This history is unfolded as soon as the intentional origins of each object are traced, and the operative and active intuitive forces are exposed; such as the operative and active intuitive forces that surfaced in the analyses of the three learning episodes.

As a result of the detailed investigation, invariable features surfaced, such as the *stereoscopic* vision of the intuition of essence, the three *genetic properties/features of intuition as an objectifying act,* or the *structures* of the *categorial intuitions* and the *intuitions of essences*. Hence the three learning episodes transcended the contexts that they appeared, the particular course, teacher and researcher, in becoming *exemplary cases of disclosed cognitive praxis*. Using a *new phenomenological description* that lends its legitimacy from Husserl's late reflections and Merleau-Ponty's radical appropriation and expansion of Husserlian ideas.

In further reflection my study is in this sense an attempt to introduce a new phenomenological perspective of intuitions and objectification, through their existential link, which is rooted in lived experience and the "realm of passive *doxa*, namely of passive believe in being" …

> [T]he realm of passive *doxa*, of passive believe in being, provides a ground of belief which is the foundation not only of every particular act of cognition and every orientation of cognition and all judgment of what exists but also of every individual judgment of value, of all practical activity bearing on what exists—therefore, it is the foundation of everything which we call 'experience' and 'having experience' in the concrete sense of the term. (1973, p. 53)

Which is how Husserl describes the essential self-evident givenness, the pregiveness that is synthesised in the "realm of passive *doxa*" (the passive stage of the Graph 3 in Part 1, § 1.5.7 and the upper 'half' of the proceeding Table 1), leading to every particular act of cognition and all judgment; which is how I understand in this research study, the main subject of the critique of original phenomenological research…

> [T]his concept of experience as self-evident givenness, of the pregiveness, first of all passive, of individual objects, has its distinctive characteristic in this: that it indicates *the fundamental structure of every experience in the concrete sense...* (Husserl, 1973, p. 53, italics in the original)

As I was participating to the sessions I was focusing on moments when the students devised mathematical objects and the origins of these objects. Probably inevitably I came across the silenced area where cognition is grounded, and started casting light on how it "affects us immediately", providing "a stimulus to action" (ibid.).[181] The exhaustive analysis and discussion have shown how the Husserlian – Merleau-Pontyian theory and methodology employed here availed me with the possibility to give an in depth theoretical and analytical account. An account that none of the theories that my study has responded in detail could have offered.

[181] Again, Husserl's emphasis on the "stimulus" rather than the "action", signifies another distinction between his—and my— approach and the Piagetian/radical constructivist ones. It is a distinction that is justified by my previous analyses of Katian/ Piagetian approaches.

PART 5. Conclusions

A. An account of the primary theoretical and analytical themes that emerge after the discussion

The aim of the study has been to disclose the role played by intuitions for the objectification process. And the theoretical, methodological and analytical layers of the study have been organised in order to make clear the structures of the central argument, which concerned the crucial link between intuition and objectification. This link is crucial since it illuminates how understanding appears in the students' consciousness, as they are involved in mathematical cognitive praxis.

Understanding through the emergence of objects in consciousness is a central theme in the Husserlian approach, which was chosen as the perspective of the study. After completing the general discussion it is time to zoom out, cashing in the study as an object, and as its parts, themselves objectified in their own sense and as parts of a whole. The laborious route of the first Part combined with the methodology/methods of the second Part which were implemented in the data analysis (Part 3), provide us with the following summary of steps:

- The introduction presented the necessary terminology, with regular reference to the theoretical (re)sources (Part 1, Ch. 1).
- The development of the theoretical approach to cognitive acts, from the *Cartesian doubt* to the Kantian distinction between *sensibility* and *knowledge,* and the instrumentalism that Husserl cancels by digging deeper into lived experience (P. 1, Ch. 1). This step opened the theoretical discussion of the distinction between the Husserlian approach that is adopted by this study and other seminal approaches that are engaged in foundational, genetic aspects of the objectification. And it showed the advantage of the Husserlian theory towards the other philosophical approaches.
- A literature example was utilised (P. 1, Ch. 3, §1.3.2), as an exemplification of:
    - the elaborated Husserlian view on 'ordinary' experience through objectification;

- the application of the (transcendental) phenomenological reduction.

Hence, a first phenomenological description of objectification was given, including the links between the reflective and the pre-reflective areas of cognitive actions, thus delving into the complexity of objectification's inaugural stages. This step introduced the complexity of students' understanding, which takes place within cognitive praxis and is consisted of reflective and pre-reflective layers.

- Husserl's theory of lived experience was introduced (Ch. 1, 4), from the genetic multiple present (the Now-moment), to the constitution of sense and meaning through objectification. Since objectification is the principal means of sense constitution in late Husserl's theory, this step was necessary in understanding the origins of objectification in lived experience; which amounts to showing how understanding is grounded on the immediate contact of the students to the world.
- Central to the study was the Husserlian approach to intuitions and objectification (P. 1, Ch. 5). Three features/properties of intuition were detected as genetically necessary due to the intentional character of conscious acts, even of the operative, embodied and pre-reflective conscious acts;[182] here is a first indication that Husserl's theory enables as a further gaze into experience, through the articulation of notions such as the lived body and its operative powers (kinaesthetic,, tactual, visual, auditory, gravitational etc.),[183] operative intentionality and intentionality of the act, categorial intuitions in two stages and intuitions of essences in three steps. The same terms returned vividly in the analysis section (Ch. 3), in each of the three students' investigations. And even other concepts like protention returned, as the—not yet fully-fledged—anticipation that operated towards Ivan's persistence in the *'asymmetrical' diagonal pattern*. And I certainly do

[182] Although it is a 'loose' sense of intentionality where the ego is not at play, consisted of the mine-ness of the experience and the arsenal of the trail of retentions and protentions.

[183] See Lappas & Spyrou (2006) for a detailed Husserlian approach of the sense of perpenticularity in the Eucledian Elements.

not aim at returning to the analysis section as I conclude; rather, I attempt to point out how actively ongoing the phenomenological reflection is, as it enables further descriptions that foresee the operative silenced aspects of conscious life, as they operate—see also the last step in this account, related to the adopted methodology. Which addresses an important aspect of mathematical thinking, since much of what goes on in it is 'silent' and it is taken for granted.

- The study also addressed the distinction between the Husserlian and the Kantian approaches to experience (P. 1, Ch. 2, 4), thus crystallising a Kantian approach to intuition that my Husserlian reading deems insufficient to go beyond a functionalistic perception of lived experience. It is a perception that still lives and thrives in constructivist research; one that does not trouble the transformation of sensibility to knowledge beyond the necessity and equal importance of both, and the hindrances that tacit models and intuitions play in acquiring correct formal understanding.
- Implications were explored, of the aforementioned distinction, to theories that are inspired by (or leave intact) Kant's perception of experience, Piaget's Kantian influence and Fischbein's Piagetian (and Kantian) influences, such as constructivist and radical constructivist inspired research (P. 1, Ch. 5).[184]
- Meno's paradox was introduced in light of a contrasting view on the aforementioned theories in mathematics education (P. 1, §1.3.1); the same paradox is employed up to the end of this study (P. 4, §4.4.4), as it exemplifies the starting point of the theoretical distinction between Husserlian and Kantian/constructivist ideas that are *crucial* for the *analysis,* and its development as the (theoretical) argument of this thesis that Husserl's theory may provide an adequate arsenal of concepts in order to deal with the learner's lived experience, hence enabling a better understanding *while*

[184] The main arguments of the Husserlian-Merleau-Pontyian framework that I adopt in this study are given in the analyses of the three cases, while the primacy of the presentation of the study's theoretical milieu—necessary for the understanding of what the research argues for—prevented further theoretical exploration of the ramifications of constructivist based research.

formulating general structures or ways of approaching the students' thinking processes.

- The aforementioned steps were part of the theoretical account that provided the following concepts and clarification of concepts emanating from the adopted Husserlian theory and methodology:
  - the living body and operative intentionality (P. 1, Ch. 4), which support and are present before a registered abstract object. The living body is co-constituted as it constitutes mathematical objects; the student's mathematical interface is in dynamic tension and prospect of expansion in any constitutive 'moment' of objectification. Operative intentionality is the key player in enacting empirical objects and reconfiguring their previous thematisations, or summoning novel ones;
  - the intentionality of act and the concomitant object constitution (the commencing of objectification) in the theory (P. 1, Ch. 3, 4, 5) and in the focused intentional analyses of the three learning episodes (Part 3). Therefore the path from the pre-reflective (passive) stage to the active, reflective one is elucidated from the Husserlian phenomenological perspective of the study (cf. table and figure in Part 1, § 1.5.7). A new phenomenological understanding of foundational, genetic aspects of objectifications is now given in the theory and in the analysis of lived investigations.
- The aforementioned concepts played a key role in identifying critical acts and their genetic features, within the students' objectification processes. And this identification was important for the empirical parts of the study, since it allowed structures of the objectification processes to surface in each of the three analysed cases, in the elaboration of the critical part that intuitions play for objectification:
  - the empirical and the abstract stages of the categorial intuitions (P. 1, Ch. 4, 5; Part 3, Ch. 3, 4) elaborated in two cases how empirical 'matter' becomes abstract mathematical objects that orient the student towards conceptualising the task-at-hand;

- the three steps of the intuition of essence (P. 1, Ch. 5; P. 3, Ch. 2, 4) demonstrated also in two cases how an empirical (Ivan's $3 \times 3, 4 \times 4$ grid pattern) or quasi-empirical (Mary's first formula) singularity is generalised and exploited mathematically;
- the three *genetic* features/properties of intuitions that allow them to be detected as such, as it was shown in the analysis of the data, in each of the three studied learning episodes. The *genetic* character of the three features/properties of *intentionality, immediacy* and the *feeling* (*filling*) *of certainty* give an original account of intuitions in general and the aforementioned intuitions in particular, without any recourse to biologistic/adaptive models; instead, the transcendental—intentional—character of consciousness is the sole principle that the whole edifice is built upon. And what was given prominent attention in this study was its critical interplay with the living body and its powers;
- clarification was provided of how *immediacy* is perceived differently by Husserl and Kant, Piaget and Kantian-based constructivism. This subtle yet critical difference was also an implicit (if not explicit) locus of my theoretical engagement related to intuition. The theoretical result concerning the aforementioned clarification and Kantian-inspired approaches contributes to the phenomenological understanding of the foundational, genetic aspects of objectifications, and they were the following:
  - Piaget and constructivism, up to its radical ramifications overcame the Kantian limitations by the recourse to *action* as the ultimate meaning endowing activity, and the acceptance of abstract intuitions. Nevertheless, the constructivist theoretical frame and its contemporary revivals repeat the Kantian approach to *experience* as already knowledge, as afforded by the structures that the categorically afforded mind will separate. The dichotomies between sensibility and knowledge, empirical and abstract intuitions—Piaget and Fischbein consider them as complementary yet strictly

separated—do not enable the understanding of the transition from empirical to abstract knowledge constitution, hence both reduce knowledge to constructions.

- Constructivist inspired studies in mathematics education research are lacking elementary instruments like intentionality, in its various expressions and forms. As a result they take the learner's lived experience for granted and they move one step further, asking how intuitions effect the learner's thinking strategies, aiming at bringing the learners in situations when intuitions are expected to operate erroneously, thus guiding them to confronting their own tacit models (examples given in Ch. 1). Which is a typical Piagetian method, based on the so called *cognitive conflict*, and it is a miniature of constructivism's perception, as it aspires to explicate issues of complex intentionalities without having explored the ground of the constitution of these forms. On the contrary, the phenomenological theory and methodology was shown capable in disclosing the operative intentional forces of the lived body, which prepared the objects before they were constituted as such, in each of the three analysed cases. It was testified in the manner of Ivan's shift of attention from symmetrical patterns to the asymmetrical diagonal pattern; it was confirmed as the drive in Diana's building up of abstraction through local algorithms, as even her last product, namely the flow chart depicted her "thought process". Finally, the living body and its operative powers were present from the beginning of Mary's home investigations, starting from the embodied take up of the diagram as the classroom bird's-eye-view. And they run through her investigation, up to her devise of the general formula due to her intuition of essence, as it was still perceived as a line that "you can stand on".

- The methodology of the phenomenological attitude was indispensable in analysing the data and enabling the application of the theory; the methods

entailed phenomenological reduction and bracketing (epochè), and their applications in successive reflections, as elements of the phenomenological unit of knowledge production (graph 1 in Part 2, § 2.2).

My thesis, more than a philosophical inquiry it is Husserlian phenomenology expanded by Merleau-Ponty's radical expansion of Husserlian concepts, and explored through practice. The aforementioned account concerns steps and concepts that were demonstrated theoretically in Part 1 and 2 of the study, and were elaborated empirically (analytically) in Part 3, in favour of a new phenomenological understanding of foundational, genetic aspects of objectification. And in the following section I will demonstrate how the aforementioned steps have contributed to the research questions and the principal findings of this study of intuitions as critical objectifying acts.

B. To what extent were the research questions answered

The phenomenological theory and methodology that is adopted by my research disclosed the operative intentional forces of the lived body, and the analysis demonstrated how the intentional forces (operative and constitutive) work together, as they prepare the objects, on the course towards their constitution as such. As we saw it in each of the three analysed cases:

- Operative and constitutive intentional forces were detected in the manner of Ivan's shift of attention from symmetrical patterns to the asymmetrical diagonal pattern, and during the three steps of his intuition of essence.
- Similar forces were confirmed as the drive in Diana's building up of abstraction through local algorithms, and even her final product (the flow chart) depicted her "thought process".
- The living body and its operative powers were present from the beginning of Mary's home investigations, starting from the embodied take up of the diagram as the classroom bird's-eye-view. And they run through her investigation, up to her devise of the general formula due to her intuition of essence, as still being a line that "you can stand on".

There is a plethora of evidence due to the data analysis, both for empirical and abstract intuitions, on how the students devised mathematical objects in order to tackle the mathematical tasks. The diversity of mathematical objects was explicated down to their *intentional origins*, in the students' lived experiences. In experiences where intuitions were the critical objectifying acts.

The intuited objects, approached as fulfilling *intentionality* with *immediacy* (and thus bringing the *filling of certainty* that the object found is the one that was sought) brought the unfolding of intuitions as objectifying acts. In each of the three students' cases we noticed how each intuition was aiming at an object, called for due to intentional forces that prefigured and prepared it as such. And some of these objects became the catalysts of each student's exploration.

The theoretical and analytical elaboration of the object's pre-objective life and its intuitive drawing to the objective state, due to operational and constitutive intentional forces, is the principal finding that this study contributes to mathematics education research. And it is precisely the finding that justifies (phenomenologically) the approach of *intuitions as critical objectifying acts,* i.e. as the crucial instigators of objectification.

It concerns a new perception of intuitions and objectification, as interrelated, and as involved in a certain yet incomplete apprehension (prehension) towards objects that are constituted for the task at hand. Which is a tension (between intuitive certainty and the indeterminacy of incomplete apprehension) that can only be resolved *through* the task; hence the stimulus for action, which is at the same time the stimulus to know the new object better.

The study suggests intuitions from this particular Husserlian perspective in which the primary, genetic features of intuitions derive from the genetic relation of the *fulfilment* of *intentionality* with *immediacy*, the latter understood as the access to the giveness of grasped objects. It is a giveness that is not taken for granted by Husserlian phenomenology. In the relentless questioning of the *how* of lived experience by Husserlian phenomenology the objects' 'giveness' is not an independent property of the object; rather it is part and parcel of the intentional

engagement of the learner with the world. And the filling of intentionality indicates a certain *feeling of certainty,* in which the intuitive conviction is expressed, that the object found is the object sought. This view that brings out *intentionality, immediacy* and the *feeling (filling) of certainty (intentionality)* as the principal properties of intuitions, with the third being the expression of the meeting of the other two in the shape of an object, not only does it give us features through which intuitions may be detected and analysed and nurtured; most importantly it reveals, after this critical questioning, an answer that is coherent and plausible, to say the least; an answer that involves intentional consciousness in its 'pregnant' state, providing us with the full menu of the intuitive process. And this happens without any resort to biologistic, adaptive behaviour, nor to any legislation for cognitive conflicts to appear. Conflicts are, instead, the lived reality of the dialectic, synthesising, operative potentiabilities of the lived body, acting as an organ of perception that cannot be boxed in the sayable, the linguistic, which is the first idealisation according to Husserl (1936). Yet it feeds and grounds the constitutive, meaningful results that encircle a certain profile, each time, an abstract object that is soon introduced to mathematical formalism. In summary, it is the better understanding of the cognitive praxis that the research argues that it has afforded us with, as the primary result of the approach of intuitions as objectifying acts.

Hence the answer to the first research question, concerning how do the students devise mathematical objects in order to tackle mathematical tasks met the answer to the last research question, concerning intuitions, in mutual response. Since the part that intuitions play in the students' commencing of objectification was answered through the exploration of the critical *objectifying* functioning of *intuitions,* in the three learning episodes.

The part that intuitions played for the constitution of mathematical objects (i.e. objectification) in each learning episode was decisive, and critical in exemplifying the texture of the materials used and of those produced, due to according intuitions. Hence it was demonstrated how intuitions and objectification are interrelated concepts, from the Husserlian viewpoint of my study. And that the

manifestation of findings was enabled due to focusing on this primordial relation, as Husserl would call it.

The manifestation of findings also justified the extensive use of Husserlian terminology, in service of the closer and fuller grasping of the open possibilities of Husserlian-based research. As my particular theme demonstrated, what sustained the cases as cases and the research as a whole was the building up of the argument concerning intuitions as objectifying acts. And the analysis of the three learning episodes was practiced so that it could achieve generality for the investigation of intuitive acts in service of objectification.

The findings of the study allow me to say that phenomenology provided me with a conceptual framework under which I could begin to organise the data that I was collecting, to decide what data for the study are, and the methods needed in order to let the data yield findings. It allowed me to acquire a bird's-eye-view of the maze that the genesis of objectification is concerned with, due to the conceptual affordances of the theory and the radical bracketing, the suspension of any judgment concerning anything but the phenomenon under investigation. This critical interrogation could be generalised as being related to putting the theme of the research and its context [185] under critical, exploratory, phenomenological questioning. And thus extracting phenomenological descriptions that can yield results with general claims—as is the case for this study—or results concerning teaching suggestions from a Husserlian perspective (Kospentaris & Spyrou, 2008; Lappas & Spyrou, 2006; Moutsios-Rentzos & Spyrou, 2014, 2015; Moutsios-Rentzos, Spyrou & Peteinara 2013, Spyrou, Moutsios-Rentzos & Triantafyllou, 2009).

The research question concerning Meno's paradox, namely how the students come to new knowledge and how they know that what they found is what *they*

[185] Such poles of interest are for example the teacher, the classroom as a unit, the effects of a curriculum and the alternative approaches to teaching and learning mathematics within a certain curriculum; also the reactivation of mathematical concepts (Lappas & Spyrou, 2006; Moutsios-Rentzos & Spyrou, 2013, 2014) and their significance for the actual involvement of the students with the historicity of mathematics in a phenomenological sense (Kospentaris, Spyrou, & Lappas, 2011Lappas & Spyrou, 2006; Moutsios-Rentzos, Spyrou & Peteinara 2013).

were looking for, was given a suggestion for an answer from the theoretical perspective of my study in the general discussion (§4.4.4). And it was further clarified in my discussion of approaches to perception in contemporary mathematics education research (§4.4.5), as my approach was contrasted to dualistic perceptions of mind and body (Cartesian) or knowledge and sensibility (Kantian), which still find ground in mathematics education.

The analysis of the students' investigations has shown that there are ways of analysing mathematical objects as phenomena of lived reality, and not as ready-made objects of our world, 'already existing' and 'ready to be transferred'. As it was shown in each learning episode, the exploration of the origins of mathematical objects in the students' lived experience, and the disclosing of the operative powers of the lived body and of the consequent constitutive acts, revealed the bottom-up process within which existing mathematical objects are reactivated (e.g. Mary's case), while allowing space for new mathematical objects to emerge. Although not any new (culturally-historically) mathematical objects appeared in the students' explorations, their intuitive investigations indicated manners in which—under different circumstances—new territories could be opened up and yield new mathematical objects.

As a mathematics education researcher, the framework that I used allowed me to give a phenomenological description of objectification as a bottom-up process, already starting in the pre-reflective ('passive') stage, due to the living body and its operative powers. This process, as it is depicted in figure 1 (Part1, § 1.5.7), enabled me to think differently about the issues of learning mathematics, and in particular of acquiring abstract knowledge through tasks. The phenomenological theoretical and methodological frames were put to the test of actual learning praxis and they managed to give results, concerning the constitutive 'moment' of objectification and the grounding role that the neglected 'passive' stage plays for objectification. My research has demonstrated that objectification can be perceived as the last step of a complex process, which is rendered visible due to the phenomenological theory and methodology. And despite the 'long path' that mathematical objects follow according to Husserl's theory, after the complex

process that was analysed here, the first steps of objectification have manifested a richness that is already illustrative of the texture and the structure of this elemental cognitive 'moment' of objectification. And the theoretical, the methodological and the analytical clarifications that have been presented have the potential to be used effectively by mathematics education researchers and practitioners (teachers) for the open exploration of the students' learning situations, and for a corresponding teaching practice.

C. Contribution to knowledge – Areas of further development

The evidence that surfaced in the analysed learning episodes, due to the phenomenological theory and methodology had a *double significance*—according to the two sets of research questions; namely one concerning the philosophical and methodological research questions, and the other one concerning learning mathematics as a cognitive praxis, from the learners' perspectives. Although the two sets of research questions and the resultant findings are inextricably linked under the rubric of objectification, I will attempt to highlight the contribution of this study to knowledge as it emanates from the double character of the research questions.

After the copious theoretical elaborations and the analytical realisations I can argue that the study has disclosed a novel phenomenological gaze on the mathematical learning experience on one hand, and has arguably advocated for a perception of *intuition* as a *critical objectifying act* on the other hand. This is in general terms the double contribution that was already announced in the preface of the study, as two *novelties* that were then expected and are now harvested. These two sets of contributions will be presented in this final section, while areas of further development will be indicated—as possibilities for expansion of the current research framework—and limitations of the study will be acknowledged.

The critical questioning of how objectification occurs made possible the illuminations that the chosen theoretical and methodological filters afforded. The study introduced many critical Husserlian terms for the understanding of the cognitive praxis, such as the living/lived body, operative intentionality and

intentionality of act, categorial intuitions and intuitions of essences. The aim of the introduction of these concepts is to facilitate an alternative approach to everyday learning situations (as was the case here), which can be useful for learning and teaching practices, and even for theoretical investigations of possibilities for bridging theories that are seemingly incompatible (see below). The reductions used in the observations, the collection and the analysis of the data, supported as they were by the bracketing of sociocultural factors and the teacher's guidance, and the minimised influence that I exercised in the students' acts, allowed the close examination of a previously unexplored *phenomenon*, namely intuitions as objectifying acts, thus bringing to the surface the stuff that objectifications are made of, to paraphrase Hamlet.

The aforementioned phenomenon concerns the cognising subject's transition from one key thought to another, the transition that takes place through embodied and lived awareness that operates and posits objects, moving towards ever more abstract entities, as it fully immerses in the mathematical formal horizons. This is the course that my research study followed, by developing the existential link of intuition and objectification theoretically, methodologically and analytically, in the necessarily limited space of the thesis.

As a result of my chosen methodology a teacher, although bracketed throughout the study[186] may now reap the fruits of her abstention and capitalise on the knowledge gained. The three steps of the intuition of essence, and the two stages of the categorial intuitions are obvious catalysts for potential reactivations of the learners' lived worlds in the classroom. I'm always tempted to introduce a concept in the class by presenting two different starting points that either meet or can be drawn by intuitive analogy. For example, I introduced the concept of the variable in the class by using the handshake problem and the Gaussian series, which 'meet' at the same formula, thus introducing the generalisation capability of variables, as independent from and integrating different mathematical situations—

[186] This is what I meant when I stressed that the bracketing does not mean exclusion or negligence of the topic/person that is bracketed, and that on the contrary, the study offers results that are useful for the teacher.

thus resembling the structure of the intuition of essence that is detected twice in this study. The same goes for the highlighting of the learners' intuiting moments by the teacher, as soon as she can detect them in the students' narrative or text. This is one main aspect of what I mean by advocating for *mathematics in the making,* which is this study's implicit theme*;* namely the re-discovery of mathematical objects such as the parabola, relations for dots in space, or integrating algorithms from empirical technology (trial-and-improvement), as was shown in Mary's, Ivan's and Diana's cases respectively.

Just as the teacher acknowledged (legitimised) every coherent mathematical practice the students devised, I never asked questions of *validity* and *generalisation* beforehand, as I tried to explore the emergence and constitution of these two concepts, as features of the learners' intentional horizons, which grounded the validations of the emerging objects and opened up their generalisable features. Neither was the objects' mathematical nature taken for granted; rather, the objects were considered mathematical due to the students' intentional horizons that endowed them with such features. Due to my phenomenological approach I neither intended to confirm or deny the 'objective' existence of the learners' objects. I instead paid heed to the appearances as intended and their ground in intuition. And as the analysis of the study has demonstrated, the establishment of the (intuitive) relation just described, a relation which fulfilment of intentionality makes possible, became the cornerstone of my study on objectification.

Yet despite the bracketing of the objective validity—thus the bracketing of what science receives as something measurable, thus culturally incorporated beforehand—both validation and generalisation returned, as soon as the phenomenological ground was set. We saw it in both intuitions of essences and in the categorial intuitions, namely how a few steps or stages respectively brought certainty of generalisations, which was confirmed (validated) and expanded—actualised formally and generalised within the mathematical frame. In other words, when the field of research is the emergence of objectification the issues of validity and generalisation occur as observed phenomena in the field, and the methodology is a solid vehicle towards transforming data sources to data, and data to findings,

firstly because it transforms them to *phenomena* in the original phenomenological sense, related with embodied intentional (transcendental) conscious actuality, rather than cogito schismatic knowledge separated from sensibility, or mythically treating the transition from one to the other—through ready-made categorial affordances.

From the practical point of view, as soon as a *researcher* and/or *practitioner* in mathematics education may reflect on the analysed episodes under the light of the adopted theory, and re-cognise or identify similar practices in her own research field or in the classroom (respectively), she will have already made a significant step towards the phenomenological attitude that I have been arguing for in this study. The very next step would be to become attentive towards the learners' drives and intentional moves. And the next step would be to involve the students' narratives and embodied significations (gestures, diagrams, metaphores) of their own perspectives in the mathematical dialogue (which would thus develop), and nurture the latter by according design—e.g. by promoting the students' enactment of their understandings. A crucial aspect in the aforementioned roughly delineated process is the dissemination and sharing between other learners, of the different approaches that some of them have, while covering all of them in time. Hence learning space is constituted and reflected upon, for each learner, enabling them to express their understandings and needs, while including their lived experiences in the mathematical discourse of the classroom.

A subtle issue concerning the application of the theory to the learning episodes needs to be clarified: as I tried to introduce all the concepts and terms that were necessary for the better grasping of the analytical methods and their significance for the appreciation of the findings as such, I might have given the impression that I merely illustrated a certain theoretical frame. Yet this interpretation would only have noticed a part of this study and certainly not the most critical one, concerning its realisation. It was rather through and due to the data that the theory was enacted/activated, and the culmination of the theory application came as the only possibility to voice the students' lived experiences, down to their intentional origins. As these origins were disclosed, after successive

reflections, Ariadne's thread was then in place, in order to bring me back to the objects as constituted, as the 'final products'. And the pathway was given shape due to further analysis and theory polishing.

Husserlian philosophy from my perspective is also particular in the sense that, more than a philosophy it is a *method of reflection* and what today is called a "research program". Contrary to the widespread portrayal of transcendental philosophy by Heidegger and others, the world of the natural attitude is not metaphysically downgraded to the status of mere appearance for Husserl, it is simply bracketed. It is put out of play for the possibility of critically important features of actual lived experience to emerge. Such as the features of *operative* and *active intentionality, immediacy* and the *feeling (filling) of certainty,* and their genetic interplay for the *objectifying* achievement of *every intuitive act*, as it was demonstrated in the previous parts of the study. These features/properties of intuition were discerned as intrinsic, genetic features, beyond psychologistic or naturalistic concessions, only after the phenomenon of objectification was viewed within the context of the learner's intentional lived experience and the operative resources of the lived body. In a similar manner the two stages of the *categorial intuitions* and the three step process of the *intuition of essence* came to the fore. Therefore the findings were shaped, inextricably linked to the theory and methodology used; and the philosophical ground that Husserl offered went hand-in-hand with a method where his concepts' implementation to our meaning endowing activities could be realised as fruitful for practical use. Therefore, the demonstration of the concordant interplay between theoretical (phenomenological) concepts and analytical practice is another contribution of my study, enabled due to critical reflections that phenomenology supported. This contribution also indicates consequences concerning areas of further development of my research, since the perception of phenomenology as a "working philosophy" and the findings that this perception enabled me are an encouraging example for further investigations within the phenomenological framework, as well as for the exploration of analogous usages of different theoretical frames, with similarly double-faceted orientations—e.g.

Vygotskian developmental experiment as a method, eliciting results under Cultural Historical Activity Theory.

A difference concerning my approach and the Kantian/constructivist one was principally clarified in the explicit use of the phenomenological methodology and of the prioritisation of the intentional potentiabilities of the lived body, as the object's intentional origins were explored, due to according bracketing. The analysis of the three learning episodes manifested through the findings how reduction and bracketing allow a close look at the phenomenon, by bracketing out what blurs the view and zoom in the critical factual necessities that Husserl's reflections have inaugurated.[187] Which is also (more precisely) a critical difference between the Husserlian–Merleau-Pontyian perception that is adopted here, and the Kantian, Piagetian, Fischbeinian perceptions of experience, object construction and intuitions. As it became explicit through the phenomenological analysis of the data, Husserlian and Husserlian-based phenomenological frames have a significant advantage in understanding the deep embodied roots of the cognitive praxis. But, although the study covered an extensive theoretical and methodological/analytical ground in the space that is available here, it has also met limitations and possibilities for expansion, which will be given an account here, as envisaged at this point in time.

The primary focus of my study has been on the individual learner from the first-person perspective, as she constitutes abstract objects in order to conceptualise the demands of the task. The more elaborated the demands, the more elaborated the intuitions that take care of them, with the path towards more elaborated abstractions remaining an open horizon for the theory adopted here. This Husserlian transcendence of Cartesian and Kantian notions of cognitive praxis renders Husserl as the pedestal and the closing act of modernity, and at the same moment inscribes (implicitly or explicitly) his cognitive problematique at the very roots of

[187] Phenomenology is approached as a "working philosophy", as a programme opened by Husserl and followed by major thinkers of our times; as I have mentioned at the preface of this study, Husserl's books are notoriously lacking in examples. The analytical details of the three learning episodes may instigate new/renewed reflections on the same or other concepts of the same thinker, relevant to our abstract field.

contemporary postmodern discussions of the same issues, with those based on Merleau-Ponty, Derrida, Heidegger, Sartre, Gadamer, Dennett, Føllesdal, Smith & MacIntyre being a noteworthy though limited presentation of Husserl's mostly unacknowledged influence. Husserlian phenomenology, as an epistemological theory and methodology, with ontic and ontological dimensions was exemplified as a valuable arsenal of theoretical/methodological approaches to lived experience, that can become powerful if polished in research practice; hence my lengthy delving into theoretical and empirical accounts of lived experience and of the powers of the living/lived body could be perceived as contributory to mathematics education research efforts that would aspire to bring theory closer to research practice.

Yet there are critical aspects of the learning experience that I have been engaged in through my study, which have remained 'untouched', since they were deliberately bracketed. They demarcate limitations of my study, coming from the upper part of the figure in Part 1, § 1.5.7, namely from the task, as a signifier of 'explicit' and 'implicit' social and cultural dimensions and motivations, such as those that activity theory is concerned with, following the dialectic materialist perception of objectification. These acknowledged issues, concerning the social and cultural dimensions of the cognitive experience were addressed only in passing in this study, due to the exclusive focus on the theme of the origins of the appearance of objectification. They were bracketed as I focused on the primordial relation of the intending act/thought and the intended object, as I was applying and testing theoretical concepts in detail. But although they are bracketed they are not neglected, since I fully acknowledge their importance for the broader understanding of the cognitive phenomenon. yet the successful application of the theory and methodology in this study invokes the need for further testing of the Husserlian phenomenology's bridging potential. In particular, the bridging potential towards the sociocultural activity theory that adopts new instruments (e.g. eye-tracker) in detecting cognitive reflective and pre-reflective actions, which is another of my research concerns, based on and extending the work presented here. Although my contribution to the joined research effort that I am engaged in is still in experimental mode, some published results are already in circulation (Shvarts & Zagorianakos,

Crossroads of phenomenology and activity theory: An analysis of the number line perception, CERME 9, TWG 17, Feb. 2015). The meaningful results that the conjunction of the two seemingly incompatible theoretical manners provides and promises is encouraging for the continuation of the aforementioned theoretical and analytical experiment.

Not yet doing but could be a dfuture project Finally, apart from possibilities for expansion of my current research framework, and limitations of my study that I can recognise, I acknowledge that my theoretical/methodological framework has its own blind spots, which I can barely sense. An indication of such a blind spot comes already from Rudolf Gachè's notion of 'groundless ground' where, echoing Derrida's origins in the history of the criticism of reflexivity he maintains that the origin and its constitutive operation are themselves situated within a syntax without origin. In this sense, the end of the presentation of my study demarcates a beginning of further theoretical and analytical investigations of the origins of mathematical learning activity and their operation towards constituting mathematical meaning.

More generally, the elaboration of theoretical terms—in carefully and persistently reflected and analysed practice—of a theory that was originally envisaged as a "working philosophy", and which developed as a methodology, could be refined into a useful instrument of meaningful analyses, for different topics within mathematics education. Since with the use of appropriate reductions and bracketing one may shift her focus from the constitutive relation of the intending act and the intended object to…

- the teacher's reflections and acts during her practice and her concomitant investigative enquiry towards alternative presentations of mathematical concepts, much in the spirit of Jo Boaler's (e.g. 1999) focusing on the importance of mistakes, struggles and persistence, while I advocate for the parallel explicit usage and strong awareness of phenomenological concepts, such as the ones presented in my study;

- the shift to the classroom as a unit, and to a given curriculum as an object, as a pole that promotes or hinders certain practices, and the gaps in it that release alternative approaches and powers, which nurture actual, lived activities and interpolations based on the latter; such a critical analysis could question the regulatory agency of the curriculum, in favour of the creative/innovative impulses of the participants' (teachers and learners) lived bodies, hence allowing alternative choices to unfold their lived truths.

Considering Husserl's relentless efforts as the starting point of my theoretical and methodological frames, the findings that were traced in the data analysis are also expected to act as exemplary cases for the theory and the methodology that enabled them. And that they are expected to be seen as a gesture for the opening of a broader dialogue concerning clarifications between theory and practice, in interpreting cognitive phenomena such as the ones that my study has dealt with in order to offer a novel perspective to mathematics education research.

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INDEX – *GLOSSARY*

INDEX OF TERMS

(with accompanying definitions and/or references to thesis pages)

A

Achievement:

> The term "achievement" can be understood both verbally, that is, as naming the process of achieving something, and nominally, that is, as naming the product of that process. The primary meaning of "achievement" for Husserl is verbal rather than nominal. Hence, an achievement is, in particular, an *intentional* achieving that constitutes or discloses an *object*. For the most part, Husserl speaks of achievement in the context of discussions of the *active syntheses* productive of categorial objects. Both the synthesizing activity and the *categorial object* are denoted by the term "achievement." The achieving is an intentional performance, the presenting or making-present of an *intentional object,* that is, the object as disclosed, as having this particular *significance.* (Drummond, 2007, p. 33)

Active synthesis:

> Active syntheses are those in which the ego functions as productively constitutive, that is, as achieving a disclosure of the object by way of subjective processes that are specifically *achievements* of the ego. Husserl's favored example of active synthesis is the *act* of judging that discloses a *state of affairs.* Also included among active syntheses, however, would be acts of practical reasoning, acts of counting or collecting, acts of multiplying or dividing, acts in which I become aware of universals, acts combining judgments in arguments or theories, and so forth. Central to the idea of active synthesis is that the ego works with 'materials' already given beforehand. For example, in the case of judging, the perceived object with its properties is the "material" articulated and synthesized in the *judgment.* In the judgmental articulation of such a perceived object with its properties, a new object—the articulated state of affairs—is constituted. Hence, active synthesis involves a 'product' (*Erzeugnis*), but this product should not be understood in the sense of a construction out of materials that are *really* inherent (*reell*) in the act itself. Instead, the ego takes pre-given, *ideal* or *ir-*

real (*irreell*) *senses* and attends to them in such a way as to fashion a new sense at a higher and more complex level. The judging that constitutes the state of affairs that, for example, the table *is* brown is founded on the perceiving that apprehends the table *as* brown, and the judgmental or propositional sense is founded on the perceptual sense of the object. (Drummond, 2007, p. 34)

Apperception:

For Husserl (2001, p. 112) apperceptions are "intuitive presentations of a co-present, for instance, the antechamber of this room, or the co-presence of alien psychic life that is given perceptually along with the alien lived-body", and he also calls them memories of the present. Also, in Husserl's words,

> "Every motivation is apperception. The emergence of a lived-experience *A* motivates the lived-experience of a *B* in the unity of a consciousness; the consciousness of *A* is equipped with an intention that points beyond, 'indicating' a coexistence. But here we must add that every unfulfilled intention, every unfulfilled horizon contains motivations, systems of motivations. It is a potentiality of motivation. When fulfillment takes place, a current motivation is there. … Fundamental for the theory of consciousness is the universal exploration of the relations of consciousness intending beyond itself (beyond its self)—what we call here apperception—to association." (Husserl, 2001, pp. 625-626)

"For Husserl, an apperception *(Apperzeption)* always presupposes and is founded on a perception (see Husserl, 1999a, § 55). To *apperceive* means to grasp something *over* and *above* what is actually perceived. … For instance, in all perception of a physical object, direct perception is of the facing side of the object, the hidden sides of the object are apperceived or appresented in an empty manner. … In his *Passive Synthesis* lectures, Husserl defines apperception as 'a consciousness of having something that is not present in the original' (Husserl, 2001, p. 367; Hua XI 234). Apperception

involves a certain awareness of properties, profiles, horizons that are not sensuously given in the perceiving itself, e.g. if I am in a room, I am aware not only of the objects that are inside the room, but also of the building in which I am. This connection between presence and absence is crucial for phenomenology. There are not only apperceptions of the things and the world but also of the self and others. Our interests, customs, convictions, judgements, etc. are grasped 'apperceptively' (Husserl, 1970a, § 59). … 'There are different levels of apperception corresponding to different layers of objective sense' (Husserl, 1999a, § 50, 111; Hua I 141). Husserl says that an apperception does not involve inference (Husserl, 1999a, § 51). For Husserl, seeing *another living body* as a subject or cogito is a typical example of an *apperception."* (Moran & Cohen, 2012, pp. 39-40).

B

Bracketing

Bracketing, or *epochè* in Husserl's terms, roughly means letting particular states of affair out of present consideration and controlling/minimising their influence, although being aware of them, in order to clear out the view of what is at stake. Since *epochè* is one of the fundamental phenomenological methods and due to the limitation of space I will employ Drummond's succinct description:

> The *epoché* is a methodological device that suspends one's participation in the *belief* characteristic of the *natural attitude,* the belief, namely, that the *world* and its *objects* exist. This suspension has its correlate in what Husserl calls the *'bracketing'* of the object, the removal of the existential index from the experienced object. The *epoché* involves, then, a *neutralization* of one's belief in the existence of the world or of an object. This neutralization might be employed in the shift from belief to doubt, or in the shift from the natural to the critical *attitude* that characterizes scientific or theoretical experiences,

or in the shift to a more narrowly characterized 'logical' or 'mathematical' attitude whose concern is solely with the deductive relations existing among different *propositions* or among objects considered purely formally, or in the shift to aesthetic awareness, or, finally, in the shift to the *phenomenological attitude.* The *epoché*, then, is in general the suspension of belief, and as such, it is a *moment* in the phenomenological or *transcendental reduction* in which one's *attention* is led back to the constituting *acts* of *consciousness* with their object simply as given and without regard to the existence or non-existence of those objects. The term *epoché*, however, when the suspension is universal, is sometimes used simply to refer to the *phenomenological reduction* itself. (Drummond, 2007, pp. 67-68, emphasis in the original)

Body-subject (or body schema):

Body-subject or *body schema* is the pre-cognitive intelligence of the body manifested through action & intertwining with the world at hand.

C

Categorial

Categorial means categorical in Husserl, but in the broadest sense, from empirical (morphological) to abstract categories, and from the subject's conceptual world in immediate experience to the scientific realm of formal knowledge.

D

E

Epoche: see *bracketing*.

Evidence (Evidenz):

Husserl's concept of *Evidenz* is variously rendered as 'inner evidence', 'self-evidence', or simply 'evidence'. Husserl understands evidence as

'an experiencing of something that is and is thus; it is precisely a mental seeing of something itself (CM § 5). It is 'nothing other than adequate self-givenness' (IP, Hua II 59). *Knowledge* in the strictest sense requires 'evidence', that is, cognitions given with *insight (Einsicht)* and with a certainty to be sharply distinguished from blind belief or a psychological feeling of conviction. For Husserl, 'the most perfect 'mark' of correctness is evidence' (LU *Pro/.* § 6). To know something is to be able to verify it by tracing it back to some evident experiences that ground it fully. Evidence can be immediate but more usually is a gradual process. As Husserl will clarify in FTL:

> Evidence ... designates that performance on the part of intentionality which consists in the giving of something itself ... The primitive mode of the giving of something itself is perception. (FTL §59)

For Husserl, self-evidence is not confined to the mathematical or logical domains. There are many different kinds of evidence, for Husserl, depending on the domain of knowledge. His standard examples of selfjustifying evident acts are our normal perceptual acts, e.g. acts of seeing which normally present the object with all the accompanying evidence necessary to warrant a judgement of the form 'I see x'. To get someone else to see requires drawing their attention to it, nothing more. Evidence is an ongoing, everyday *'achievement' (Leistung)* in all cognitions where the object is given in a satisfactory form, with 'intuitive fullness' *(anschauliche Fulle)* or as Husserl prefers to say, in which the object gives itself. Husserl emphasizes that self-evidence involves the transition from empty intentions to fulfilled ones (a *process* in the sense of *'fulfilment'),* that it is the ('absolute') 'self-givenness' of the object (itself), so that it's the *mode* of the givenness of an object, that it's something that belongs to the *form* of our acts. Husserl also discusses *Evidenz* in his *Formal and Transcendental Logic*

(§§ 105-1 07) and *Cartesian Meditations* § 6. (Moran & Cohen, 2012, pp. 114-115)

Husserl (2001, pp. 108-109) mentions evidence a number of times, since it is crucial in building up his theory of *perception* and ultimately *epistemic knowledge,* as (both) grounded on *intuition*, which is in its turn indissolubly linked to the *evidence* brought up by self-giveness.:

> A datum that is given in immanent perception, i.e., that is adequately given in each Now does not therefore admit of any further confirmation with respect to this Now. Still, it does occur as a fulfillment insofar as the preceding perceptual phase already points to what is to come. This fulfillment is a fulfillment of an anticipation and is a definitive, absolute fulfillment, or evidence.

In other words, evidence appears already as "a fulfillment of an anticipation", at the primordial phase of lived-experience, as the fulfillment of the protention of the just-having-been-Now, thus been deprived of metaphysical features that are still eminent in Kantian transcendental philosophy, due to the distinction between sensibility and knowledge.

The primordial phase of lived-experience is the *flow of appearances,* and the according trails of *retentions, primal impressions* and *protentions*. It is the *field* where the temporalisation of our lived-experiences takes place in inner-time consciousness, and the *ground* for the constitution of objects and objectivities, in passive and active syntheses, coloured by the discipline due to which they gain validation, as they serve the ever renewed task horizons, and as they become the carriers of reactivation and of renewal of knowledge.

The profile of the real that is absorbed from the lived-body, the subject-body, is intuitively grasped; and here is detected Husserl's primordial sense of *intuition,* equivalent to mere, immediate—instant and unmediated—contact. The feature of immediacy is kept throughout Husserl's descriptions, for all forms of intuition.

F

G

Giveness

> something is said to be given insofar as the ego yields to the allure and has turned toward it attentively, laying hold of it in egoic acts of interest, cognition, explication, or examination, etc. (Husserl, 2001, p. xlii, from Anthony Steinbock's introduction)
>
> cf. *self-giveness* below

H

Horizons

- Inner horizons

  The inner horizon—by virtue of the three-fold structure of the living present—intentionally refers to other appearances or presentations of the identical intended object. For example, the momentary perceptual phase in vision presents one side or aspect of the object from a particular perspective. This momentary phase, however, has intentional connections to past presentations of other sides of this same object, and it has intentional connections as well to possible presentations of the object that might arise in the course of a continued perceptual inspection. This noetic structure allows one to say from a noematic standpoint that the genuinely and directly appearing side refers beyond itself to its other sides and aspects. The inner horizon, in other words, unites in a single awareness a multiplicity of differentiated senses internal to the total significance the identical object has for us. The experience including its inner horizons thereby presents the object that is the identity proper to these unified senses. (Drummond, 2007, pp. 96-97)

- Outer horizons

> The outer horizon, on the other hand, intentionally refers to other objects in the "surroundings" of the object that is the thematic concern of consciousness. These surroundings may be a spatial field—a background of other objects—in and against which the perceived object stands out. But it might also be, say, a context for a judgment, a context, for example, comprising relevant judgments about similar objects or comprising other judgments belonging to a theory in which the present judgment will take its place. At the most general level, the outer horizon comprises intentions presenting other objects that co-inhabit the world with the intended object. This view of outer horizons also has its noematic counterpart in the view that the world is the ultimate horizon of all intended objectivities. (Drummond, 2007, p. 97)

Hyle, hyletic data or sensuous contents:

> *hyle,* ancient Greek term for matter. Aristotle brought the word into use in philosophy by contrast with the term for form, and as designating one of the four causes. By *hyle* Aristotle usually means 'that out of which something has been made', but he can also mean by it 'that which has form'. In Aristotelian philosophy *hyle* is sometimes also identified with potentiality and with substrate. Neoplatonists identified *hyle* with the receptacle of Plato. (Audi, 1999, p. 408)

> Hyletic data include, first, the sensuous contents that present the objective, sensible determinations of an *object*. This is the fundamental meaning for Husserl of the expressions 'sensuous contents' and 'hyletic data." However, hyletic data also include, second, *feelings* such as sensuous pleasures and pains that are involved in the awareness of the *value* of objects. And they include, third, what Husserl calls "drives," the instinctual tendencies that involve bodily feelings of certain kinds. Husserl understands all such hyletic data to be *real (reell) moments* of experience.

Husserl isolates the basic notion of hyletic data in reflecting on the *perception (Wahrnehmung)* of *material things* in space. He imaginatively varies the perception such that the sensible qualities of the *intended object remain* constant while their *appearance* to us varies. Husserl attributes this change in appearance to changes in the *fullness* and vivacity of the really *(reell)* inherent sensuous contents. He concludes, therefore, that the *intentional* experience must be composed of two real *(reell)* moments: an intentional *apprehension* or noesis and the sensuous contents. The *noesis* is a *form (μορφή)* that animates or interprets the sensuous matter *(ὕλη).* The basic idea is that the hyletic data are the presenting or representing "stuff" that is really inherent in the *experience*. However, because hyletic data are sensuous in character, Husserl extends the scope of the term to include all really inherent sensuous moments. Hyletic data do not themselves bear the mark of *intentionality*; they are referred to an object only by virtue of their being intentionally "formed" by the apprehension.

Originally Husserl thought that all acts have some sort of material stuff or hyletic data to be intentionally formed, but his analyses of both *inner time-consciousness* and of *categorial acts* persuaded him otherwise. Consequently, it appears that he retained the doctrine of hyletic data only for the impressional *moment*, that is, *primal impression*, within the *momentary phase* of consciousness. Moreover, although he initially characterized hyletic data as a really inherent *moment* of the experience, there are places in Husserl's works where he speaks of hyletic data more noematically as the immediate sensible *presence* of the objective determination itself. Finally, the broadest sense in which Husserl speaks of hyletic data is to refer that which is passively pregiven as the materials on which active thinking operates. (Drummond, 2007, pp. 97-98)

I

Ideation:

Husserl modifies the view that we grasp the species through abstraction and instead claims that we have an act of *ideation,* an essential intuiting of the species themselves (see Husserl, 1970b, § 88). 'Eidetic intuition' replaces 'ideation' in *Ideas* I, although the term does continue to appear in later writings (Moran, & Cohen, 2012 pp. 2, 26).

Intentionality:

> Intentionality is an essential peculiarity of the sphere of mental processes taken universally in so far as all mental processes in some manner or other share in it. … Intentionality is what characterizes *consciousness* in the pregnant sense and which, at the same time, justifies designating the whole stream of mental processes as the stream of consciousness and as the unity of *one* consciousness. … Under intentionality we understand the own peculiarity of mental processes "to be consciousness *of* something." … a perceiving is a perceiving of something, perhaps a physical thing; a judging is a judging of a predicatively formed affair-complex; valuing of a predicatively formed value-complex; a wishing of a predicatively formed wish-complex; and so forth. Acting bears upon action. Doing bears upon the deed, loving bears upon the loved one, being glad bears upon the gladsome; and so forth. In every actional cogito a radiating "regard" is directed from the pure Ego to the "object" of the consciousness-correlate in question, to the physical thing, to the affair-complex, etc., and effects the very different kinds of consciousness *of* it." (Husserl, *Ideas I,* pp. 199, 200)

Husserl uses the words 'intention' and 'intending' throughout his works in a way similar to our daily usage, indicating a purposeful striving. Intentional consciousness is, in all its forms, focused on finding satisfaction in the intuited self-having of lived experience. Consciousness wants to go toward evidence; that is what forms its goal, its telos. In this sense, all conscious life—as Husserl would say

in his later work—rests under the rule of a 'teleology.' (Held in Welton, p. 14)

In this regard we speak of the 'intersubjective constitution' *[intersubjektiven Konstitution]* of the world, meaning by this the total system of manners of givenness, however hidden, and also of modes of validity *[Geltungsmodi]* for egos; through this constitution, if we systematically uncover it, the world as it is for us becomes understandable as a structure of meaning [*Sinngebilde*] formed out of elementary intentionalities The being of these intentionalities themselves is nothing but one meaning formation operating together with another, 'constituting' new meaning through synthesis. And meaning is never anything but meaning in modes of validity, that is, as related to intending ego-subjects which effect validity. Intentionality is the tide which stands for the only actual and genuine way of explaining, making intelligible. (Husserl, 1970a, p.168)

[A]s soon as one has progressed far enough in the reorientation of the epochè [it should be clear] to see the purely subjective in its own self-enclosed pure context *as intentionality* and to recognize it as the function of forming *ontic* meaning (Husserl, 1970a, p. 169, emphasis added)

Phenomenology begins from the intentional relation between constituting subjectivity and its correlated constituted object. Intentionality points to the intrinsic correlation between the object as meant and the subjective act which apprehends or means ('intends') it. … Husserl speaks of the need to go back to the 'intentional origins' and attempt to follow the build-up of 'sense-formations' which we eventually experience in a completely immediate way as the whole intuited life-world, understood as an 'integrated framework of meaning' (*Sinnzusammenhang,* C 284; K 331) or 'meaning formation' *(Sinnbildung,* C 378; K 386). (Moran, 2012, p. 53)

> Husserl took over Brentano's conception of intentionality but greatly expanded and clarified it. He felt that Brentano's own conception was too deeply embedded in an internalist and introspectionist outlook. Brentano maintained (at least in his earlier years) that psychological states revealed themselves directly and immediately to consciousness (their *esse* is *percipi)* and that they were exactly as they presented themselves, whereas the external world was known only indirectly. … Husserl records that Brentano never accepted the analyses in the *Logical Investigations* as a development of descriptive psychology and the 'mature execution' of Brentano's idea (*Phen. Psych.* § 3, p. 24; Hua IX 34). Husserl, then, must present intentionality in an entirely new context, stripped of the misleading apparatus of modern representationalist philosophy. For Husserl, intentionality properly construed made visible the manner in which things present themselves to consciousness in their various modes of meaningfulness. (Moran, 2012, pp. 54-55)

Intrapsychically:

> Intrapsychically is rendered here as within the conscious activity of the individual. The term does not imply a process cut off from the subject's world-as-lived, and it involves intersubjectivity in two fundamental ways:
>
> - The materials that are involved in the intrapsychic moments of consciousness are culturally and socially mediated.
> - The very transition from prereflective to reflective intrapsychic acts maintains intersubjective intentional functions, which translate qualities into relations that are transmittable to others, thus in awareness of the subject's community and its norms—e.g. mathematical norms, i.e. manners (treatments) that when they are applied they transform objects to admissible abstract objects. In other words intrapsychic acts realise the

shifting[188] and they intentionally (intersubjective intentionality) introduce the new concept to the mathematical field of knowledge, from the very beginning.

L

Lifeworld:

> [T]he lifeworld … is always already there, existing in advance for us, the —ground of all praxis whether theoretical or extratheoretical. The world is pregiven to us… not occasionally but always and necessarily as the universal field of all actual and possible praxis, as horizon. To live is always to live-in-certainty-of-the-world. (Husserl, 1970a, p. 142)
>
> The lifeworld is the latent, normally unexamined givenness of experiences, situations, events, etc.. The lifeworld typically goes forward without self-conscious intervention or purposive design; it incorporates an unnoticed, unprompted expectedness. Only when some taken-for-granted aspect of a lifeworld shifts does the lifeworld become apparent (Seamon, 2013), and such a negation—which still takes place in the lifeworld and will fall back out of awareness once the break will be incorporated—takes place when the learner intuits an object, due to the reconfiguration of the horizons that the constitution of the new object entails.
>
> And in a 'rough' sense, objectification gives the phenomenologist a unique chance to lay bare the lifeworld's taken-for-grantedness itself.

Lived body:

> "For embodied consciousness Merleau-Ponty, like Husserl, reserves the term "living body"" (Tito, M. J., 1990, Logic in the Husserlian context, p. 185). Husserl uses the term *Leib,* which has been translated both as *'living body'* (e.g. Husserl, 1970a, pp. 50 (n. 15), 106, 161, 217-218, 324, 331) and *'lived*

[188] The shifting is meant as a new embedding of the object's horizons, a shifting of horizons, in Husserl's terms.

body' (e.g. Husserl, 2001, pp. 19, 50, 73, 265). I have adopted the '*lived body'* terminology since it seemed more faithful to other Husserlian terminological choices (e.g. *lived experience* instead of *living experience* – e.g.. Husserl, 1973, p. 321) and since it is one used even in classical translations (Husserl, 2001). The same term—*lived body*—is also used by renowned scholars for both Husserl and Merleau-Ponty, e.g. Donn Welton (1999b) and Catherine Morris (2012). Morris (p. 49) also informs us that "[p]rior to Sartre and Merleau-Ponty, Husserl had already made a crucial distinction between 'the lived body' *(Leib)* and 'the body-object' *(Korper)*". My adaptation of the '*lived-body'* terminology is also concomitant to Moran's expression *body as lived* (e.g. Moran, D., 2010, p. 135). Finally, cf. "Romdenh-Romluc's claim that for Merleau-Ponty 'the body is a form of consciousness'" (Morris, K., 2012, Ch. 3, fn. 2), and "[for Husserl] the essence of consciousness is to be embodied" (Tito, 1990, p. 187).

Lived experience (Erlebnis)

Lived experience is a key concept that Husserl introduced in order to "lay bare the 'sources' from which the basic concepts and ideal laws of *pure* logic 'flow', and back to which they must once more be traced, so as to give them all the 'clearness and distinctness' needed for an understanding, and for an epistemological critique, of pure logic" (LU Introduction § 1, **I,** p. 166; Hua XIX/1 6-7). Looking back in 1925, Husserl described the aim of the *Logical Investigations* as follows:

> In the year 1900-01 appeared my Logical Investigations which were the results of my ten year long efforts to clarify the Idea of pure Logic by going back to the *sense-bestowing* or cognitive achievements being effected in the complex of *lived experiences* of logical thinking. (Husserl, 1977, p. 14)

Husserl introduces *lived experience* as a field from which we can take distance from, in the phenomenological attitude, approaching it as a subjective gesture that constantly has the potential to 're-invent the world',

the lived-world of the subject in the first place. He expresses the investigation of lived experience as a new task, which marks his "return to the things themselves. The new task consists in…

> the attempt to go back radically and consistently from the respective categories of objectivities and ask about the modes of consciousness determinately belonging to them, about the subjective acts, act-structures, foundations of lived experience, in which objectivities of such a character become objects of consciousness, and above all become evidently self-given. (Husserl, 1977, p. 20)

It is another manifestation of Husserl's critique of rationality, as he maneuvers his conception of evident, intuitive self-giveness to the core of the validation of the sciences, logic and mathematics included. The objects of consciousness "become evidently self-given" in what Husserl calls manners of giveness. Studying the manners that objects are given in the students' consciousnesses, through the students' lived experiences, is a significant part of phenomenological—or phenomenologically inspired—research since, as Held explains…

> [M]anners of givenness are the way in which intentional consciousness carries out its lived experiences, but in the same stroke these are appearances-of-something; in other words, they are manners of self-revelation, of existing things presenting themselves. If we were to apply the Cartesian question, asking whether we should add these lived experiences to the category of consciousness's 'outer' or 'inner' world, then we could not appropriately comprehend their richness. Lived experiences break down this dualism; they are the In-between, that which originally opens the dimension of intentional appearance within which consciousness and the world have already met—before any subject-object rift. (Held, in Welton, 2003, p. 28)

Hence Husserl is positioned beyond the Cartesian dualism by his allowance of "existing things presenting themselves". He also moved beyond Kant's distinction

between sensibility and knowledge (*Critique of Pure Reason*), since embodied, passive object generation holds the primal cogitata, the proto-objects of thought according to Husserl.

M

Manners of giveness and lived experiences:

> [M]anners of givenness are the way in which intentional consciousness carries out its lived experiences, but in the same stroke these are appearances-of-something; in other words, they are manners of self-revelation, of existing things presenting themselves. If we were to apply the Cartesian question, asking whether we should add these lived experiences to the category of consciousness's 'outer' or 'inner' world, then we could not appropriately comprehend their richness. Lived experiences break down this dualism; they are the In-between, that which originally opens the dimension of intentional appearance within which consciousness and the world have already met—before any subject-object rift. (Held, in Welton, 2003, p. 28)

Meaning: see *sense*

N

Noetic:

> Noetic process is a process directed by the *ego pole* towards the *object pole,* having the specific intentionality of objectivation, i.e. the intentionality to release *objective sense* out of transcendental and immanent objects that appear in the Life-world of the subject.

O

Originaliter:

The term ‘originaliter’ is used by Dorion Cairns and Husserl in order to express the ‘grasping’ in its 'personal' actuality; another term used by Husserl for the same reason is *originarily* (cf. Moran & Cohen, 2012, p. 264).

Object-like formation:

Objectlike formation [*Gegenstiindlichkeit],* that is, something that exhibits the basic structure of an object (including an object-phase), but is more "elementary" than an object in the full-fledged sense or has *not* (yet) exhibited objectivity. This is not to say that the objectlike formation cannot have its own internal continuity maintained in passive time-consciousness, but only that it is not the result of active processes that give it an identity such that it becomes a theme of cognitive interest. [fn 21:] This is why I translate the term *Gegenstiindlichkeit* as objectlike formation, rather than objectivity or even objecthood. The expression *gegenstiindlich* I render "objectlike," and depending upon the context, "with objects," since Husserl sometimes uses the expression to indicate it as qualifying the noun: e.g., *gegenstiindliche Feld.* Here he does not mean a field that is like an object, but rather, a field filled with objects that can potentially become thematic. (Husserl, 2001, p. xli, in Anthony Steinbock’s introduction)

P

Passive genesis:

See passive stage.

Passive stage or passivity:

Passive stage is for Husserl the pre-reflective level of experience, the level of 'preconstitution' (sic). “The most fundamental form of passivity is the flow of temporal experience whereby *retentions* and *protentions* just occur as part of the experiencing of the present. Moments of *time* and moments of sensuous experience (e.g. apprehending a colour patch as having a uniform colour) are united together by *association”* (Moran & Cohen, 2012, p. 237).

The passive stage involves *passive genesis,* which "names those processes which give the world its pregiven, stable and harmonious character. It also gives the objects encountered in the world their sense character that is encountered as fully formed by active perceiving etc." (Moran & Cohen, 2012, p. 28). One law of passive genesis is that every experience becomes a trace in retention and does not vanish completely (see Husserl, 2001, p. 114); and its universal principle is *association* (ibid.).

The following summation is written by Anthony Steinbock in his preface to the 2001 edition of *Analyses concerning Passive and Active syntheses* (Husserl, 2001) and it serves as a concise description of what *passivity* is for Husserl:

> Passivity designates a dimension of experience that a regressive inquiry into origins unearths, and which serves as a point of departure for an explicative genetic account that traces motivations leading back "up" to cognitive activity. More particularly, passivity is that realm in which, through fundamental laws of association, affective forces spur an egoic attentiveness to objects, enabling acts of remembering and expectation to constitute objects as such, i.e., as in-themselves-for-us.

Pregivenness:

According to the *Analyses,* something is *pregiven* insofar as it exercises an affective allure on me without being grasped by me as such, responsively or egoically. [fn 20:] This is certainly one sense of pregivenness. Another sense also occurring around the time of the *Analyses* is what is always already there for the individual who can in turn appropriate it either passively or actively. (Husserl, 2001, p. xlii)

Presentification

Presentification is a term used by Husserl in order to distinguish objects of regular perception, which are given directly and with 'in the flesh' here-and-now presence (presentation), from objects that do not have this in-the-flesh

givenness (re-presentation) (Moran & Cohen, 2012, pp. 260-261). Anthony Steinbock (Husserl, 2001, p. 110, fn. 64), gives us a subtle distinction between presentification and re-presentation:

> Empathy *[Einfiihlung]* as a mode of presentification does not make present a previous or futural perceptual object because the other or the alien can in principle never be given 'originally.' This is certainly different from a remembering that literally re-presents its object, or from a futural presentification that anticipates a futural object as present, or finally from a co-presentification in which something that is not given in the original at present, but can in principle be present (cf. below 367, 373f., 377f.). Accordingly, rather than translate *Vergegenwärtigung* as re-presentation, which would be well-suited for temporal acts, I prefer the common neologism, presentification, since it includes the full complement of acts ranging from imagination to empathy. (Husserl, 2001, p. 110, translator's footnote)

Presentification and re-presentation need to be distinguished from representation (which is a key term for constructivist research), since not only do objects of consciousness additionally bear temporal features that are pre-constituted and their history is registered (e.g. in recollection the awareness of their past feature), but their intentional history (e.g. in perceptual acts that are recollected and even in futural presentifications that "can in principle be present") is also retrieved, in gradations of clarity and fusion. The associative character of consciousness guarantees that presentifications and re-presentations are not arbitrary or just reflexive. Reflective and pre-reflective motivations—intentionalities in the phenomenological language—play a crucial part (e.g. in the retrieval of previous presentations). In these motivations takes place a fundamental interplay between subject and object, between the intending and the intended. On one 'side' (the *object-pole* in Cartesian terms – c.f. Husserl, 1970a, p. 207) the objects' allure, calling for signification the ever searching torchlight of consciousness on the other 'side' (the *object-pole* in Cartesian

terms); and most importantly, the embodied operations of the living body that synthesise these profiles of *lived reality* (which is the ground within which the aforementioned dualism of ego and object poles is cancelled, precisely due to these operative syntheses), thus preparing them for possible objectification, if chosen for particular drives that seek fulfilment. It is under this light—the light that phenomenology affords us with and at the same stroke allows us to explore tasks as this one that my study touches—that lived experience and the operations that emanate from it, i.e. the texture of our life-world, make possible—and in a sense determine—our consequent object constitutions.

Primal impression

For Husserl, every temporal experience has a moment that he calls the 'primal or primordial impression' (Urimpression). In early works, he sometimes refers to it as 'primordial sensation' (Urempfindung, Hua X 324). Husserl describes this primal impression as the moment of creation (Hua X, 105); it is the very core of the living present. However, it is a necessary eidetic law that this primal impression must be modified into a retention. The primal impression can be said to found the retention, yet the primal impression as such can appear only in the retention. There is no absolute experience of the primal impression as such. (Moran & Cohen, 212, p. 262)

Protention

Protention is that phase within the *momentary phase* of *consciousness* or *living present* that intends yet-to-come phases of *absolute consciousness.* (Drummond, 2007, p. 171)

The now moment, retention and protention are three mutually related, non-independent parts of each conscious *lived experience* according to Husserl's analyses of *time consciousness*. According to Ideas I § 77, a protention is the 'precise counterpart' of a *retention*. Just as retention is not yet memory, so protention is not yet anticipation in the full sense which is a form of *presentification*. The protention modifies the already elapsed retention.

> Husserl speaks in this regard of a backward streaming or backward mirroring (Rückstrahlung) of the protention in the retention. Retentions motivate protentions and protentions are founded on retentions. Protentions and retentions belong to passive experience. (Moran & Cohen, 2012, p. 265-266, emphasis in the original)
>
> [P]rotention is not yet the fully fledged conscious act of anticipation but a structural component of any *lived experience (Erlebnis).* (Moran & Cohen, 2012, p. 207)

Psychologism:

> 'Psychologism' entered the English language as a translation of the German word '*Psychologismus*', a term coined by the Hegelian Johann Eduard Erdmann in 1870 to critically characterize the philosophical position of Eduard Beneke (Erdmann, 1870). The relationship between logic and psychology was fought over most intensely in the German-speaking lands between 1890 and 1914. This dispute centered on the question whether logic (and epistemology) are parts of psychology. Gottlob Frege and Edmund Husserl are the best-known figures of this controversy. John Stuart Mill exerted great influence in both sides of the controversy, since his *System of Logic* (1843) was not only a key inspiration behind much German-speaking psychologistic philosophy, but it also contained some crucially important anti-psychologistic ideas (Godden, 2005). Most German-speaking philosophers, from the 1880s onwards, agreed that the following arguments deserved the label 'psychologistic' (I shall write *PA* for 'psychologistic argument'):

(*PA 1*) 1. Psychology is defined as the science which studies *all* (*kinds of*) laws of thought.

2. Logic is a field of inquiry which studies *a subset of* all laws of thought.

Ergo, logic is a part of psychology.

(*PA 2*) 1. Normative-prescriptive disciplines — disciplines that tell us what we ought to do — must be based upon descriptive-explanatory sciences.

2. Logic is a normative-prescriptive discipline concerning human thinking.
3. There is only one science which qualifies as constituting the descriptive-explanatory foundation for logic: empirical psychology.
Ergo, logic must be based upon psychology.

(*PA 3*) 1. Logic is the theory of judgments, concepts, and inferences.
2. Judgments, concepts, and inferences are human mental entities.
3. All human mental entities fall within the domain of psychology.
Ergo, logic is a part of psychology.

(*PA 4*) 1. The touchstone of logical truth is the feeling of self-evidence.
2. The feeling of self-evidence is a human mental experience.
Ergo, logic is about a human mental experience — and thus a part of psychology.

(*PA 5*) 1. We cannot conceive of alternative logics.
2. The limits of conceivability are mental limits. *Ergo*, logic is relative to the thinking of the human species; and this thinking is studied by psychology.

Who actually held these views, indeed whether anyone did, was hotly contested at the time, but it seems reasonable to attribute *PA 1* to Theodor Lipps (1893) and Gerardus Heymans (1894, 1905), *PA 2* to Wilhelm Wundt (1880/83), *PA 3* to Wilhelm Jerusalem (1905) and Christoph Sigwart (1921), *PA 4* to Theodor Elsenhans (1897), and *PA 5* to Benno Erdmann (1892). (Kusch, 2011, *The Stanford Encyclopedia of Philosophy*)

R

Reflection / self-reflection:

'Reflection' (*Besinnung)* and 'self-reflection' (*Selbstbesinnung*) are key terms for Husserl. … Husserl distinguishes between *Besinnung* and *Reflexion* ('reflection'):

> The reflection *[Besinnung]* in question is a particular case of that self-reflection [*Selbstbesinnung]* in which man as a person seeks to reflect upon the ultimate sense of his existence. We must distinguish between a broader and a narrower sense of self-reflection: pure ego-reflection [*Ich-Reflexion*] and reflection upon the whole life of the ego as ego; and reflection *[Besinnung]* in the pregnant sense of enquiring back into the sense or teleological essence of the ego. (C 392n; K 510-11 n.1)

Husserl, then, understands *Besinnung* more as a kind of existential self-meditation rather than as a Cartesian-style introspection, even one reflecting on one's life as a whole. … Husserl uses several variations on the notion of 'reflection'. He frequently characterizes his reflection as a 'backwards reflection' (*Rűckbesinnung,* C 17; K 16) or questioning back' (*Rűckfragen,* or *Zurűckfragen,* cf. C 56; K 57, C 69; K 70; see also K 185), a regressive enquiry into the 'original motivation' (*Ursprungsmotivation,* C 57; K 58) that gave rise to modernity. This concept of reflecting or questioning back is central to what he calls 'genetic' or 'genetic-historical' enquiry. (Moran, 2012, pp. 49-50)

Reflective level of the experience

Retention (also cf. protention)

Perception takes place in the now, in the present, and its object is apprehended as immediately present, 'in the flesh' *(leibhaftig,* as Husserl says), as being there in the same temporal phase as the mental process itself. In memory, however, while the *lived experience (Erlebnis)* is in the present, the object remembered is not experienced as being in the present, but precisely as not present and as 'having been'. Memory suffers from an essential *inadequacy* in that things can be represented that were not, in fact, ever perceived (see *Ideas* I § 141), or different memories can be fused into one memory. For Husserl, *retention* (or, in earlier terminology, 'primary memory', Hua II 67) is not yet memory in the strong sense ('secondary memory"), although it forms the basis or ground for both passive and active

rememberings. Rememberings present objects as whole entities, whereas a retention is a part of a perceptual awareness, it is a 'just past' that is still there in a reduced or modified sense. It still has a kind of 'impressionality'. (Moran & Cohen, 2012, p. 207)

Reduction:

Husserl's example on and explication of the Phenomenological reduction:

> Now, if I take as my point of departure this reflection that remains completely unphenomenological, this "I have heard the song," I can now, as phenomenologist, carry out the reduction on it, bracketing the spatio-temporal mundane actuality of song and singer. Then I will gain the transcendental phenomenon, namely, the past transcendental phenomenon of my—of the ego's—earlier acoustical perception of the song, whereby the real song itself is only the intentional object of the hearing.
>
> In this way I can gain the entire realm of my memories as phenomenologically reduced, thus the realm of all memories of objectively mundane things and processes, and then of all memories in general, e.g., memories of mathematical proofs that I have carried out; and I gain them not only as present facts, but according to their remembered intentional content. And what finally results from this is my (i.e., the Ego's) past transcendental life with all of its past poles which, insofar as they are object poles, are bracketed, while the ego that is everywhere identical is the transcendental ego, and should not be bracketed, just as little as the past transcendental lived-experiences.
>
> One can also put it in this way: In the case of rememberings and with all other presentifications that we will take up later, we deviate from our earlier principle of putting out of play all positings that are carried out in the lived-experience itself. I only put out of play the positing, the memorial belief in the past objectivity, but not the belief implied in it, namely, the belief in my past ego and my past

> lived-experiencing, and my past perceiving in which my past life was given perceptually. We emphasize this because the transcendental subjective element that is past is also precisely transcendentally subjective, and because we initially want to appropriate transcendental subjectivity as a whole in one stroke as far as the unity of the ego and its lived-experiences can reach (regardless of whether they be present or past). In doing this we follow the evidence that itself lies, in part, in the phenomenological reflection on the present, in part, in the phenomenological reflection on the past (that is, in the reflection penetrating into the intentional content of memories). But we do not ask whether or not this evidence is apodictic, whether or not it is better than the evidence, e.g., of external perception (that we had to put out of play to begin philosophically).
>
> We can treat memories of the future, expectations, in the same way as memories of the past. (Husserl, 2001, pp. 453-454)

Husserl claims to have discovered the phenomenological reduction some time around 1905 in his Seefelder manuscripts; and its discovery marks a sharp break between the descriptive psychology of the *Logical Investigations* and the transcendental phenomenology of the mature Husserl (Moran & Cohen, 2012, p. 107). The phenomenological reduction aims to overcome the naive thinking about the object in the nature attitude as something that simply exists on its own and comes to understand the object as correlated with a specific mode of apprehending it. The perceived object, as intentional correlate of the perception, is distinct from the real object (Moran & Cohen, 2012, p. 75).

Reminiscent of the universal Cartesian doubt, it [the methodological technique of the phenomenological reduction] is nevertheless different therefrom. Whereas the distinguishing characteristic of Cartesian doubt is that it annuls the positing of an object's existence or the validity of a judgment, the distinguishing characteristic of the phenomenological

reduction is that it refuses to understand this annulment as the opposite of the positing of the existence of objects and the general validity of experience that characterizes our natural experience—a positing Husserl calls the "general thesis of the natural attitude" (Hua 3, §30). The phenomenological reduction, in other words, is not the negation of the general positing characteristic of our ordinary experience. The content is not negated, but our affirmation is withheld. In the performance of the phenomenological reduction, we attempt to call the universal positing characteristic of ordinary experience into question, to hold it reflectively before ourselves as a positing whose validity is to be examined. Our participation in the affirmation characteristic of ordinary experience is suspended, and the objectivities given in experience are not lost to our reflection but are instead considered only as presumed existents. They remain available for reflection just insofar as they are experienced; the index attaching to them, however, has changed, and their status as objects of experience has been modified so that they are now viewed exclusively in their being as objects of that experience in which they are posited. It is not, therefore, as it was for Descartes, the object that is disconnected in the performance of the reduction; it is the philosopher's participation in the positings that characterize the ordinary experiences of the natural attitude. The reduction is a change in attitude that *leads* our attention *back* to the subjective achievements in which the object as experienced is disclosed in a determinate manner and to the achievements in which we realize the evidence appropriate to confirming or disconfirming our natural experiences. These achievements have a certain kind of priority over the object that they disclose in a determinate manner, and the investigation of them reveals how it is that we come to experience the objects in those determinate manners; how our different experiences are related to one another; and, therefore, how the different kinds and levels of objectivity are related; and, finally, how our experience confirms or disconfirms in fulfilling intentions what was merely emptily intended or mistakenly intended. (Drummond, 2007)

Re-presentation: cf. Presentification

S

Self-giveness:

Anthony Steinbock notes, in his introduction to the 2001 translation of Husserl's *Analyses concerning Passive and Active Syntheses*:

> The more common expressions like *Selbstgebegenheit* (with which Husserl means both the giving of the self of the object on the part of the subject as a noetic process, and the self-giving of the self of the object from the object), I render as *"self-givenness"*; in this case, one should hear in the expression "self," then, not the subject, but the ipseity of the object, the self-givenness of object it-self in the intentional relation. As such, the object it-self is never experienced in a neutral manner; it exercises an affectively significant allure on the perceiver to be constituted as such, that is, for the "ego" to turn toward it attentively and to constitute it as a theme of interest in an active manner. The fact that *something* is actually heard or seen or smelled, etc., is due to "affective rays" radiating from the object, drawing in its wake the horizonal referential implications. (Husserl, 2001, p. lvi)

Husserl himself mentions self-giveness a number of times, since it is crucial in building up his theory of perception and ultimately knowledge, as grounded (both) on intuition, which is in its turn indissolubly linked to self-giveness. We discern here (Husserl, 2001, p. 108) Husserl's particular realism, binding the empirical (in a fully transcendental sense) with the transcendental, and allowing a further view, than Kant's perspective on experience. Husserl's distinction between the Kantian and his transcendental version is also tangible where he denies the inaccessibility of the object's in-itself (which is what self-giveness is all about, according to Husserl), an

intangibility instituted by Kant. In its place, Husserl brings the *indeterminacy* of this knowledge, which cannot be fully fulfilled, since it *"is never determined definitively":*

> To be sure, we know that even a perception, in particular, an external, transcendent perception, can occur in syntheses of fulfillment and not only as a perception confirming an intention; rather, it can even occur as a mere intention that becomes fulfilled in new perceptions. This happens, for example, when we perceive a tree from the front, and wanting to know it better, draw nearer to it and now perceive it in new perceptions; by determining the tree more closely, we also have a fulfilling confirmation. Meanwhile, every external perception harbors its inner and outer horizons, regardless the extent to which perception has the character of self-giving; this is to say, it is a consciousness that simultaneously points beyond its own content. In its fullness it simultaneously points into an emptiness that would only now convey a new perception. *The self-givenness of a spatial thing is the self-givenness of a perspectival appearing object that is given as the same in the fulfilling synthesis of appearances intertwining and devolving upon one another.* But it is the same object that itself appears now this time in one way, now another time in another way, appearing in other perspectives, always pointing from a perspective to ever new perspectives in which the same object that is exhibited is continually determined more closely, *and yet is never determined definitively.* For we always expect appearances of newly opened, empty horizons. Thus, where there is no horizon, where· there are no empty intentions, there is likewise no [synthesis of] fulfillment.

Sense ('making sense' of experience):

The notions of 'sense' *(Sinn)* and 'meaning' *{Bedeutung)* are central to phenomenology. Phenomenology is concerned with *meaning,* but one must be careful here not to think solely of linguistic meaning. Husserl tends not to

distinguish sharply 'sense' *(Sinn)* from 'meaning' (*Bedeutung),* although he is aware of Frege's distinction. Insofar as he does make a distinction, he tends to employ 'meaning' primarily for linguistic meaning. The term 'sense', for Husserl, carries wider connotations, in that non-linguistic activities, such as perceiving, remembering and so on, also involve 'sense', i.e. we perceive the paper lying on the desk as having its own existence, self-identity, spatio-temporal continuity, objecthood, and relation to and distinctness from other objects, but also its own history, cultural meaning, significance and so on.

The central focus of phenomenology, it can even be said, then, is the problem of *sense, of meaning.* Husserl often couples together the concepts of 'sense' and 'validity' (*Sinn und Geltung,* e.g. C 7*6;* K 78) that things, people, situations, social actions and so on have for us as experiencing subjects in the world. From the standpoint of phenomenology, a thing's ontological status cannot be distinguished from its sense or meaning. Something can be a religious icon in one cultural context and a cultural adornment in another (e.g. a tattoo). Hence Husserl speaks of 'being-sense' (*Seinssinn*), or 'ontic sense' (Carr's translation, see C 122; K 124), or ontic meaning' (C 100; K 103), referring to this interwovenness of ontological standing and significance. He also regularly speaks of something's 'validity of being' or ontic validity' (*Seinsgeltung*, C 77; K 79).

All experiences, no matter how vague and apparently inconsequential (even illusions, hallucinations, dreams, reveries), make sense in some form, and the kind of sense an experience conveys has its own particular mode of 'givenness', its own peculiar way of coming to prominence, its own temporal duration, its structural form, implied connection with other experiences and so on. To perceive something as a physical, material thing, for instance, involves many levels of constitution of sense, but to see it as also a picture or *artwork* is to grasp it in a further and quite distinct mode of meaning disclosure, and distinct again from a tool used for a practical purpose, or a relic approached through religious veneration, a souvenir and so on. Language articulates this rich differentiation of kinds of object, but our rich

perception already intuitively makes these discriminations, and it is phenomenology's task to document them and grasp their essential character.

For Husserl, sense is not simply something outside us that we apprehend, it is something that is 'constituted' or put together by us due to our particular attitudes, presuppositions, background beliefs, values, historical horizons and so on. In short, phenomenology is a reflection on the manner in which things come to gain the kind of *sense* they have for us. (Moran, 2012, pp. 50, 51, 52)

State of affairs:

'State of affairs' denotes the way things stand. States of affairs are those realities (loosely speaking) that we refer to by propositions such as *'S* is P' or *'A* and *B* stand in relation *R'.* If the state of affairs is as it is asserted to be, then the proposition is true. In this connection, the proposition is 'the truth-bearer' and the state of affairs is 'the truth-maker'. Or, put the other way around, the state of affairs is what we typically 'express' in perceptual propositions; e.g. 'the candle is on the dining table'. The assertions that usually count as 'knowledge' concern *states of affairs',* e.g. the moon is full; 20 million people live in Australia; Napoleon was defeated at Waterloo; blue colours are called 'blue' in English; this is a book; A is not ~A; the global economy is becoming increasingly integrated. In short, our *knowledge* of the world concerns *states of affairs,* not entities *simpliciter (LI* I, Proleg. §6). (Russell, p. 114, emphasis in the original)

T

Telos, teleology:

The term 'teleology' (from the Greek *telos, τέλος*, meaning 'purpose' or 'goal') means goal directedness, being directed to goals or ends. … Husserl uses the term 'teleology', especially in his later works, to refer to the specific networks of ends that motivate human life and culture. *Intentionality* as directedness has an inbuilt teleological character, for Husserl, e.g. *empty intentions* aim

at *fulfilment*. Husserl specifically talks about the 'teleology' of western culture in the Crisis and associated texts (e.g. *Vienna Lecture*). (Drummond, 2007, p. 314)

Transcendental ego:

The transcendental *ego* is *transcendental consciousness* in its subjective dimension and as reflected upon. The transcendental ego is the *intentional* center of all conscious life and, hence, of all objectifying *experiences*, all affects, all *valuations*, and all *volitions* and actions. Husserl's transcendental ego differs from *Immanuel Kant*'s insofar as it is not a formal identity accompanying all experiences; instead, for Husserl the transcendental ego is a self-transforming identity over *time*. It is an identity by virtue of its self-unification in the *living present* and *inner time-consciousness*, and it is self-transforming insofar as it acquires new convictions and new *habitualities.* In this way, the transcendental ego is self-constituting, and it also discloses itself as a *psychological ego* in the *world.* Husserl sometimes uses "transcendental ego" in a wide sense equivalent to "transcendental consciousness". (Drummond, 2007, p. 205)

On the subject of the ego, see Lester Embree's chapter (in Carr & Casey, 1973), where he devotes careful attention to Husserl's texts in order to discover just what Husserl meant by this perplexing term.

Transcendental reduction: see Reduction

APPENDIX

Appendix A. My recently published article in Educational Studies in Mathematics

The article that was published in January 2015 in Educational Studies in Mathematics (ESM) is the result of two years of research and refinement of the research results, thus containing a substantial part of the theoretical, methodological and analytical work that has been exposed in the five Parts that comprise the body of the thesis. Anna Shvarts, my co-author, helped me substantially in cutting down the text within the deadline, and supported me when I needed to complete the work, while teaching full-time in the complicated working conditions of the economic crisis that has hit my country the last years.

I want to thank the anonymous reviewers of ESM for their deep involvement with the ideas of the article, and their open attitude towards the theory, the methodology and the analysis involved in it, judging that the work is worthy of publication to this seminal journal in our field. In particular, I want to thank

- Reviewer 1, who acknowledged the clear and informative presentation of my theoretical presentation and of my Husserlian/Merleau-Pontyian application of the theory to the data, despite his personal reservations; "think[ing] that by wanting to see a more primitive empirical intuition at work in Mary's thinking, the authors miss the 'sedimented' character of her starting point". Unfortunately, my focus on giving a phenomenological description of the reactivation of the parabola that took place in Mary's case, did not allow me to point out explicitly how this very process reveals the sedimentation that mathematical objects undertake, and the reactivation process as actual de-sedimentation. Allusions are given in the preceding chapters (Part 3, §3.4.2.4) of Mary's sedimented understanding of the formula as a collection

of points that have a certain property, although beyond the scope of this study.

- Reviewer 3, who acknowledged that
  - the Husserlian and Merleau-Pontyian position is contrasted explicitly with the Kantian and the differences are clearly demonstrated;
  - that the analysis of the student's developing awareness, as it is enabled by the deeply drawn theory, is novel;
  - and that the implications for researchers and teachers are insightful and applicable.

The role of intuition in the process of objectification of mathematical phenomena from a Husserlian perspective: A case study.

Andonis Zagorianakos · Anna Shvarts

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Abstract The research is a study of the Husserlian approach to intuition, informed by Merleau-Ponty's theory of perception, in the case of a prospective teacher of mathematics. It explores the two major stages-categories of intuition, the essential relations between them, and their vital role in the emergence of empirical and abstract mathematical objects, as they are used by the student in order to conceptualise mathematical phenomena. The student's activity is analysed to its intuitive origins, and an intuition of essences manifests its significance for generalisations and insights for mathematical proofs. Through an in depth phenomenological data analysis, intuition is delineated as an essential mediator between the learner's world-as-lived and her objectification process. Finally, some implications for teaching and learning are suggested.

Key words: Intuition; Husserl; Merleau-Ponty; Phenomenology; Embodied consciousness; Living body; Operative intentionality; Intentionality of Act; Objectification; Objectivation; Constitution of an object; Mathematical objects.

1. Rationale of the research – Aims

Anyone engaged more or less in mathematical enquiry knows that the results of this endeavour are usually presented as stress free, as an almost 'natural' process. In the textbooks, mathematics is ordered in a linear way and presented as a unity in the sense of a ready-made product (Steinbring, 1991). Teachers present theorems to the students unaware of the etymology of the word, which is associated with the

final view (thea) at the top of a hill or a mountain, after the laborious climbing has taken place. A similar phenomenon takes place when teachers encounter their students' thoughts and suggestions, since the teachers gauge them according to their perception of mathematics and mathematical knowledge, neglecting the particular paths that their students take and their *key ideas*. The qualification of these key ideas in order to become visible is the target core of our present research. In the analysis of the data we will shed light on the *figuration process* of the learner's world-as-lived in relation to mathematical knowledge; and trace the origins of learning in the reactivation of a cultural object, a parabola, through equidistance from a wall and a fixed point, first actually then formally, by a prospective teacher of mathematics, due to successful (fulfilled) intuitions. Intuitions became the origins of such *key ideas* under the Husserlian phenomenological lens that we are drawing from, as objectifying acts that signal the immediate access of the individual to her lived world, from which she acquires the texture and the material of her thoughts. Using a corresponding phenomenological data analysis our current research has a four-fold aim:

- To accentuate the *given* aspect of learning mathematics as a non-prepackaged experience of the learner's world–as–lived.
- To manifest how intuition may be realised as an essential objectifying act, operating as the link between formal mathematical knowledge and the learner's lived-experience.
- To explore and to exemplify the role of different kinds of intuitive acts in the first stages of objectification. To attempt an inquiry into the 'things themselves', the learner's lived-experience that becomes the material for empirical and abstract (mathematical) objects, and to explore essential actual and structural relations between these objects.
- To suggest possible implications of a neo-Husserlian phenomenological perspective for teaching practice.

2. Kantian and Husserlian approaches to objectification and intuition – A new perspective to intuitions as objectifying acts

2.1 A critical difference between Kantian and Husserlian approaches

Most definitions and classifications of intuition are endowed with a *constructivist* lens originating in Kant, Descartes or Plato, especially when it comes to abstract intuitions. Processes that inhabit the *cogito,* the discursive human mind, take over, and the abstract intuitive processes are ultimately settled in internal, relational cognitive models. The mature Husserlian approach on experience (after his Time Lectures and Ideas II)[189] that our research adopts, cannot be properly understood as a meta-Kantian philosophy without a space permitting clarification of its crucial yet highly neglected divergence from Kant's notion of *experience* and *knowledge*. "Experience is an empirical knowledge," writes Kant (cited in Tito, 1990, p. 78). By experience Kant means *objective* thinking, *knowledge.* In other words, there is a predisposed categorial network that the subject is stored with, which makes perception possible. Although the issue exceeds the limits of this paper it may suffice to say that the difference between Husserl and Kant is very subtle yet crucial, since Kant recognises experience as *already* endowed with objectivity, while Husserl considers objectivity itself being *constituted* by the subject(s) (as a major pole of constitution), the communities and the social-cultural-historical entities, while the scientific communities are also at the centre of his interest (1970a).

Husserl introduces "a new phenomenological attitude which focuses on the correlation of experienced and experiencing within lived experience itself as a foundation of the sciences" (Husserl, 1983b, p. ix). In a necessarily crude sense we could say that for Kant, the objects that are constructed by the subject are 'separations' of experience by thinking, which is already categorial;[190] while for Husserl "[t]he object is the idea of the fulfilled sense (as fulfilled completely)" (Husserl, 2001, p. 449). Which leads him to the essential "how the 'objective' a priori is grounded in the 'subjective-relative' a priori of the life-world or how, for example, mathematical self-evidence has its source of meaning and source of

[189] Husserl, 1991 and 1989 respectively, in the references.

[190] This issue also amounts to the difference between Kant's representation and Husserl's re-presentation, which exceeds the space limitations of this paper.

legitimacy in the self-evidence of the life-world" (1970a, p. 140). Kant's persistent categorial membrane separates the subject from the world by making impossible her immediate contact with it, while Husserl introduces the immediacy of perception and objects as "self-given in a simple intuition" (Husserl, 1973, p. 150, §12). This immediate contact with the life-world (not the world of Kantian things-in-themselves) takes place through the living body, which is also constituted at the moment of object constitution. The embodied nature of consciousness and the immediacy of contact yield proto-knowledge directly from the real, thus allowing Husserlian phenomenology to overcome previous varieties of phenomenological ideas such as the Kantian one, where phenomena are reduced to mere opinions (Radford, 2014).

Kant's principle idea that the transcendental ego is already installed with some qualities *beforehand* (*a priori*) remains as a *genetic* seal, inherited as we shall see by the constructivist followers of Kantian ideas in mathematics education and in particular intuition. Moreover, Kant's concept of intuitions does not anticipate any abstract intuitions, apart from the a priori intuitions of time and space. And these intuitions yield knowledge due to their ideality and not due to their immediacy (Hintikka, 1972, p. 344). Husserl's approach is in sharp contrast to Kant's approach, since he argues for the constitution of all categories through his concept of *categorial* intuitions (section 2.4).

2.2 Husserl's notion of objectification

The concepts and theory used in this paper are drawing on Husserl's theory of objectification/objectivation, as it is explicated in his mature work and posthumously published manuscripts (Husserl, 1970a, 1973, 1983a, 1983b, 1991, 2001), as well as the use of Husserlian ideas (operative intentionality, living body) by Merleau-Ponty (1964, 2002a, 2002b).

Husserl delineates a complete course of the objectification process in five stages (Husserl,, 1936 pp. 163-165), from the "intrapsychically constituted structure" (p. 163) of the *proto-geometer,* to the final, ideal stage of the object, through the social scale of "the community of scientists as a community of

knowledge living in the unity of a common responsibility" (ibid. p. 169). During the *first stage* an object appears in front of consciousness, "in original 'self-evidence'" (p. 163). In the second stage the same object appears and it is identified as the same. Then, in the third stage the subject has isolated an idea of object and applies it to different situations. In Husserl's 4th and 5th stage of the objectification the idea obtains the status of an "absolute ideal Objectivity" (Derrida, cited in Moutsios-Rentzos, Spyrou, & Peteinara, 2014, p. 30).

Husserl's unexplored work on objectification that concerns us here is related to the first stage of objectification, "the history of the object itself as the object of a possible knowledge" (Husserl, 2001, p. 634). Husserl introduced *objectivation*—his special term for the early stages of objectification—of objects, from their objectlike to their object status (2001, part 3). The object appears for the first time in consciousness as a separation from its hyletic (raw) empirical material, by fulfilling the intentionality that was aiming at it. The fulfilment of intentionality leads to the *constitution of the object*, namely to a novel thematisation, a unity of previously latent features that perception brings together as a profile of the subject's lived reality, a new configuration from the 1st person perspective.

The acts of *constitution* don't just appear *in a vacuum,* since:

- The individual involved in the primal constitution of a (mathematical) object for herself is already in a world, which houses and provides the material for the learner's intuitions.
- The subject herself has pre-reflective operative intentionality to render her proto-objects intelligible for others right from the start (Husserl, 1970a, §54b) and then objects appear in front of her consciousness as existent for others.
- There is a meaningful tension in the interplay between the subject that exercises the objectifying act, and the object that is intended (Kolen, 2005), a negotiation of meaning through the task/activity.

In an *educational context,* the key ideas that were involved in the objects' genesis are 'awaiting' their reactivation, for a meaningful sense of the teaching/learning praxis, and the students are guided through the unfolding of

hundreds and thousands of years of evolution that each mathematical object conceals in its refined, 'final', ready-made mode in which it is presented. This aspect is stressed in the cultural-historical approach towards objectification, which is introduced as the process when "students gradually become acquainted with historically constituted cultural significations and forms of reasoning and action" (Roth & Radford, 2011, p 48). In this approach a new object appears in subjective consciousness as a result of learning, by the "transformation of cultural objective knowledge into an object of consciousness" (Radford 2013, p. 25).

The important problem here is that the cultural objects are originally unknown for the learner: "How will you set about looking for that thing, the nature of which is totally unknown to you? Which, among the things you do not know, is the one which you propose to look for?" (Merleau-Ponty, 2002, p. 431). From the phenomenological perspective "perceptual consciousness is not an empty box, into which a perceptual object shows up unannounced and ready-made" (Husserl, 2001, p. 606), but due to the "continual coinciding of sense" (ibid.) new objects appear. If from the cultural-historical perspective to objectification a student is supposed to be able to reactivate culturally predisposed objects, for phenomenology it is the constitution of new objects that allows the learner to approach and to enrich the cultural objects, through her lived experience. And it is the *history of objectivation* (2001, p. 634) that reveals the origins of the *reactivation* of mathematical objects and objectivities in the acts of new constitutions, regardless if the latter have already entered the mathematical domain or are *illuminations* (Hadamard, 1954) that could achieve the final stages of the objectification that Husserl (1936) delineates.

At the moment of constitution the learner manifests herself as the Husserlian *transcendental subject* "that intentionally engages the world and discloses its significance" (Drummond, 2007, p. 204). The constitutive moment reveals both, the constitutive subject and the constituted object, through "the correlative intentionalities of which they are the poles, through whose function they have, and have attained, their ontic meaning" (Husserl, 1970a, p. 182). The intentional identification of the object by the objectifying act, performed with the *immediacy* of

self-giveness is intuition itself, which will be at the centre of our investigation in this paper.

2.3 Two kinds of intuition and the categorical structure of knowledge

Intuition is a highly controversial topic in science and philosophy and some authors even suggest that the concept of intuition may be misleading and that it should be avoided in scientific reasoning (Fischbein, 1999, p. 11). Many researches describe different kinds of intuitions that are supposed to function on different levels of cognition, from empirical to abstract knowledge. Piaget distinguishes *empirical* intuition—which concerns the physical or psychological properties of objects (Beth & Piaget, 1966, p. 224)—and *operational* intuition that "possesses an immanent logic" (ibid., 212). Fischbein (1994) classifies intuition by their origins as *primary* and *secondary*, where "Primary intuitions refer to those cognitive beliefs which develop in individuals independently of any systematic instruction as an effect of their personal experience. (p. 64), while "the category of secondary intuitions implies the assumption that new intuitions, with no natural roots, may be developed. Such intuitions are not produced by the natural, normal experience of an individual" (ibid., p. 68).

Let us look more closely to the philosophical base of Piaget and Fischbein's positions, since the latter's approach is also endowed with a *constructivist* lens, originating from the Kantian theory of mind (Otte, 1998). For Fishbein intuition is an act of consciousness that presents empirical or mental objects to us, based on pictorial models, ideality, abstractness, absolute perfection and universality (Fischbein, 1994, 2001). In this approach intuition appears in an already structured, thematised world, as a "combination of incompleteness of information and intrinsic certitude" (Fishbein p. 51). The 'blindness' of constructivism to the pre-objective stage of the cognitive lived experience makes the constitution of abstract entities incomprehensible: it leaves the structure of the lived world unexplicated and the origins of constitution are lost in predisposed categories. Such a case appears in recent mathematics education research (Andra & Santi, 2013), where "[i]ntuitive thinking can be seen as the sensuous side of intellectual activity" (p. 31). Verifying

the Kantian tradition of Fischbein in classifying intuitions as separated—in "sensuous" and "intellectual" classes— 'inescapably' informs an instrumentalist view on cognition, which supposes that "what we are, feel, think" is *constituted* by artifacts (p. 30).

"For Husserl there is a level of experience that has not yet been subjected to the objective categories, a level of experience that is the ground of the objective categories. The 'I' is such an experience, an experience that is preobjective, and is not knowledge" (Tito, 1990, pp. 78-79). Husserl (1970b, 1983b) gave a privileged position to intuition for systematic knowledge (Smith, 2007), by placing it at the core of his *principle of all principles* (Husserl, 1983a, p. 44). He approached intuition as the *objectifying* act in which a person grasps something immediately in its bodily presence and also as a primordially given act upon which all of the rest is to be founded. For Husserl the source of intuitions is reality itself, constituted by the subject not in a metaphysical manner but through the immediate contact with it, the intuitive contact. "What is given in intuition for Husserl are components of reality, not our impressions of them, … at this absolutely crucial point he remains totally foreign to the Kantian way of thinking" (Hintikka, 2003, p. 174).

2.4 Husserl's Stages of intuitions – Empirical and Abstract intuitions

According to Husserl, the intuitive process starts from the separation of the forms that are imposed on the raw empirical material (the *hyletic* data or simply *hyle,* deriving from the Greek word for matter), from the data themselves.

"Empirical intuitions are due to reality impinging on my consciousness, not to the activities of my own mind. … In order to reach empirical intuitions, I have so to speak to look outwards, not inwards" (Hintikka, 2003, p. 178). Husserl calls this first stage *Anschauung*—intuition—and the outcome of this stage is an empirical object. For Husserl the empirical intuitions are not an *experience* in the Kantian sense, since consciousness is already embodied. The separation is not performed due to the categories that manipulate the empirical (sensory) data; nor is it a mere mental recognition of patterns that are so far familiar only to others. It is intuition that allows embodied intentional consciousness to distinguish objectlike entities from raw

material and transform them into empirical objects. The objectilike entities reside consciousness until operative intentionality will call them forth to a new thematisation, a new unity that is constitutive for the new object. Embodied consciousness is what Husserl and Merleau-Ponty call the *living body,* "which is never absent from the perceptual field" (Husserl, 1970a, p. 106, also cf. Merleau-Ponty, 2002a, p. 87, and Tito, 1990, p. 185). The body[191] plays an essential role in the fulfillment of *operative* intentionality, since it is "a sensory-motor behavior through which the world is constituted for man (sic) as the world of human consciousness" (Merleau-Ponty, 1964, p. xvi). The *operative intentionality* is "already at work before any positing or any judgment" (Merleau-Ponty, 2002a, p. 498) and it "produces the natural and antepredicative unity of the world and of our life" (Merleau-Ponty, 2002a, p. xx). It is pre-reflective but this does not mean that it is unconscious (Reuter, 1999, p. 76); and it can be revealed:

> We do not see the operation, since it operates. We see what would be missing without it; we circumscribe it as what makes speech be a "speaking of" ... and not be the conscious having of the idealities implied in speech. (Merleau-Ponty, 2002b, p. 44)[192]

Due to the operative intentionality the cognising subject does not *construct* objects (by *imposing* categories on experience) but rather *enacts* and *inhabits* them, by separating them from the unstructured raw material. Despite the immediate feeling of certainty that intuition brings there is a tension, since the intuitive moment with all its fullness of certainty is realised as a *prehension,* i.e. the "inadequate or imperfect grasp of something, where the content of the grasp adumbrates or points to something beyond what is given … a form of incomplete cognitive apprehension" (Tragesser, 1977, pp. 17-18). What is crucial from our cognitive point of view is that the operative intentionality makes possible the *intentionality of act* (Merleau-Ponty, 2002a, p. 486), "which is that of our judgements and of those occasions when we voluntarily take up a position" (p. xx). The intentionality of act is responsible for the

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[191] The body as (intentional) subject, not as an object (cf. Reuter, 1999, pp. 71-72).

[192] The extract is from Merleau-Ponty's notes on Husserl's *Origin of Geometry*.

"more complete apprehension of the object" (Tragesser, 1977, p. 19) that is achieved through its application to the task.

The other crucial novelty of Husserl's approach towards Kantian epistemologies that concerns us here is his introduction of the *categorial intuitions*, which are intuitions of *abstract* objects (Husserl, 1970b). The categorial intuitions take place in two stages, following empirical ones and using their products.

When the second intuitive stage takes place—the *categorial* stage—an abstract object will emerge, in a process that is "*more like seeing than imagining*" (Hintikka, 2003, p. 177). According to Husserl:

> …Husserl's essences, do not come to us separated from the empirical objects whose "forms" they are. … This intuition has the effect of opening to my consciousness, not only an empirical object consisting of matter and form, but that form or essence in itself, separated from its *hyle.* (Hintikka, 2003, pp. 177, 180)

The *intuitions of essences (Wesensschau)* are a special case of *categorial* intuitions and we refer to both of them as *abstract intuitions*. For the same reasons *Wesensschau* will refer to both of them.[193] Let us sum up the two main kinds of intuitions according to Husserl:

- The *empirical intuition (Anschauung)* enables the subject to extract forms-objects from the raw empirical material (the hyle) through her lived experience, due to operative intentionality. The disentanglement of forms turns them into empirical objects and they become known as such, as soon as they are separated.
- The *abstract intuition (categorial, Wesensschau)* enables the subject to extract abstract forms-objects from the products of empirical intuition, as well as from the results of more elaborated objectifying acts. The disentanglement of abstract forms takes place due to the intentionality of

[193] Examples of a categorial intuition and an intuition of essences are given in sections 5.3.2, 5.3.4 respectively.

act, and turns them into abstract objects that are still based on the subject's lived experience.

2.5 Intuitions as interrelated forms, functioning in a similar way

The stages of the categorial intuition, i.e. the empirical and the abstract stages are linked, although the second one may immediately succeed or take place long after the first one.

Since the empirical objects of the first stage can become materials for the second one that produces abstract objects, Husserl provides a strong indication for the emergence of abstract objects as *interrelated with empirical objects,* which either are or they are related to empirical material, previously objectified (section 5.3.2). And he approaches cognition as linked to the learner's *embodied consciousness*, rather than to *pre-existing objective categories.* He grounds the seeing of 'essences' on an embodied perception of conscious activity, and identifies intuition as the operator "on the overlap of my consciousness and reality" (Hintikka, 1995, p. 82), as an actual mediator between the world-as-lived by the individual and her constitution of abstract mathematical objects. His inquiry concerns "the way in which the life-world constantly functions as subsoil, into how its manifold prelogical validities act as grounds for the logical ones, for theoretical truths" (Husserl, 1970a, p. 124).

Categorial intuitions follow the pattern of empirical intuitions: The latter intend objectlike entities and by intuitive synthesis thematise them into empirical objects; the former collect invariable features (essences) of empirical objects and they synthesise them into abstract objects.

Husserl's theory elucidates how an empirical object is transmuted to an abstract object during the 2nd stage-category of intuition, how an abstract object undergoes the same process as it is transformed into another abstract object and so forth. Husserl, as a mathematician, is aware of the particular nature of *mathematical* norms and *mathematical* objects which can proceed higher and higher in a process of abstraction.

2.6 Criteria for intuitive moments.

To identify a particular act as intuitive we used the following criteria of *intentionality, immediacy* and the *feeling of certainty*:

- *Intentionality* is the property that makes intuition an objectifying act. In empirical intuition it is a straightforward (immediate) effect, and it is determined as such by Husserl in abstract intuitions: "what makes seeing an essence an intuition is not that it is seeing an *essence,* but that it is seeing an *object* which is 'itself given'" (Hintikka, 2003).
- *Immediacy* is the intuitions' critical feature, which distinguishes them from other concepts such as memory or imagination (Husserl, 1970b; Hintikka, 2003; Held, 2003). Immediacy is present in both intuitive instances and it brings with it a *feeling of certainty.*

These criteria were used in the analysis of the empirical data allowing us to understand the different kinds of intuition that the students used, according to their relations to the students' lived experience. The Husserlian theory of intuitions offered a solid explication of their origins in the cultural and time dependent world of experience, and it shed light on their expansion to more elaborated intuitions.
Being attentive to visual and kinesthetic evidences of the *sensory-motor* pre-objective experience of the student we expect to have a glimpse on how intuition transforms raw material into objects, thus contributing something critical for mathematical education by better understanding the constitution of the mathematical objects by the students.

3. The description of the course - The case of Mary

The data analysis is a *case study* from a course that took place in the academic year 2010-2011 and involved a group of 13 students, training to be teachers in British secondary schools. The course consisted of twenty 3-hour sessions, where the main targets were the production of generalisations by the students and their diverse conceptualisations of mathematical phenomena, such as the embodied understanding of curves. The teacher avoided the provision of information concerning the tasks, intervening in the students' methods or interrupting their discourse (as described by Brown, 2012); he only described the operational aspect of the tasks. For this reason this course was particularly appropriate for our research

where we intended to look at the learning process from the first person perspective and to bracket (i.e. set aside from the present consideration) influences of cultural tools and practices, which inevitably accompany any education.[194]

The students were invited to use their common sense and mathematise their results in any way they thought appropriate. They were working in small groups of 3 to 5 people of their own choice and it was suggested to continue ther investigations at home and push them as far as they could in order to submitt their findings in their coursework.

In the episode that will be analysed[195] the students were asked to make sense of a curve through exercises centred on the students' bodily movement. The teacher of the course asked the students "to think how they can move whilst remaining equidistant" from the wall (10 foot-paces) and a student who remained stationary. No other direction or indication on how to proceed was given to the students, who were free to handle the activities any way they thought suitable.

The student of our *case study,* Mary (pseudonym), was embodying the fixed point, while another student was acting out the curve (fig. 1). The first author participated in Mary's group while avoiding to facilitate the students' investigations. A few days after the session Mary presented an elaborate drawing (fig. 2) and a detailed account of her treatment of the activity at home. At the end of the course she submitted her report on this activity as her favourite amongst several.

The session was audio and video taped (after gaining the students' permission) and Mary's home work was analyzed through interviews and the text of her coursework. An interview was conducted a few days before the student delivered her coursework, then an additional interview took place and an e-mail was used in order to clarify a few issues. In the following sections we analyse Mary's

[194] It doesn't mean that we suppose intuitive processes to be limited to individual processes: we acknowledge that for the fullest understanding of intuitive processes in the learning practice attention needs also to be paid to the social-cultural dimensions, as it is attempted in Andrà & Santi, 2013, Andrà & Liljedahl, 2014.

[195] A different perspective of this session appears at a paper that was published recently (Brown, Heywood, Solomon, & Zagorianakos, 2013).

data, as they were triangulated or cross-checked from the aforementioned data sources.

Figure 1: The other student with her back to the wall and looking at Mary (the fixed point) moves her hands perpendicularly towards the wall.

4. Qualitative methodology of the data analysis

The driving force of this study is our intention to explore the genetic moments of cognitive activity, in the sense that Husserl (1936, 1989, 1973, 1991, 2001) gives to it and Merleau-Ponty develops it to the primacy of perception. We aimed to describe the mathematical phenomena as they occurred in the tasks from the student's individual perspective, and to start from a viewpoint free from hypotheses about what the student's constructions should be and as free as possible from preconceptions (Husserl, 1970b). For this reason we attempted to apply a phenomenological reduction that focuses on the experience in which the object is given to the subject, instead of analysing the object of experience itself. Under this perspective the consequence of bracketing was necessary in order to approach the student's objects *as they appeared*. Starting from 'bracketing' the body of the mathematical tradition and 'bracketing' cultural interventions in the learning process we were driven towards "*bracketing* of questions regarding the actual existence of the experiences on which we reflect and the object of those experiences and focusing instead on the object just in the manner of its *appearance*" (Drummond, 2007, p.160). The first author participated in the whole course of learning sessions and he adopted the teacher's strategy; he inevitably was not a detached and impartial observer but an internal participant who tried to enter into the perceptions of the students and see the learning situations as they saw them. It was an ongoing and adjusting process of bracketing and coming to the phenomenological attitude (Moustakas, 1994; Lester, 1999). In this context the current study approaches the

learning process from a point of presence within and through it, attempting to make it as transparent as possible by accessing its origins in the student's conscious acts, deeply rooted as they are in the individual and common horizons of the world-as-lived and perceived by the learner.

Under the perspective that was adopted during the participation in the sessions, the student's communicable expressions (verbal, figurative, diagrammatic, etc.) were explored on the different levels of systematisation, generalisation and abstraction, trying to gain access to the "intrapsychically constituted structures" (Husserl, 1936, p.163) from which these expressions originated. Through these expressions we came to intuitions as a main mechanism of appearance of the objects. By applying the phenomenological attitude, we argue that the treatment of the particular case study could gain generality towards mechanisms of objectification not by inductive conclusion but by phenomenological evidence that "covers other acts besides simply perceptive seeing", while "[s]ense-formations whose nature it is to exist as subjectively produced results are 'grasped' originaliter in being produced" (Dorion Cairns cited in Husserl, 1936, p. 13). It is in this sense that we achieved evidences of different kinds of intuitions, which were revealed not as empirically existent and observable, but as the residue of phenomenological analysis.

5. The three stages of the student's investigation

In further analysing the empirical data our phenomenological method allowed essential structures to be revealed by acknowledging the learner's intentional transcendental ego as an objectifying unit. In the course of close exploration of all the material three stages emerged from Mary's investigation, namely the *embodied* (when she was in the classroom) the *diagrammatic* and the *formal.*

5.1 The embodied first stage of Mary's investigation

The analysis of the audio and video recordings of the first stage revealed important clues about the curve and her formal negotiation that followed:

- Mary, who acted as the fixed point, thought that the curve was a straight line parallel to the wall, midway between the wall and the fixed point, while the

other student, who acted as the curve, thought that it was a semicircle with a radius of 5 foot-paces and its centre located at the fixed point. She was moving to different positions—curve points—with her back to the wall and looking at Mary, repeatedly moving her hands perpendicularly towards the wall (fig.1),[196] in order to indicate each curve point's distance from it.

- The other student and the first author, in order to show that the curve could not be a straight line parallel to the wall, used the right triangle formed by the fixed point, the midpoint—located on the straight line segment midway between the fixed point and the wall—and another point on the line parallel to the wall from the midpoint.
- The students could not reach an agreement on what the curve would look like, so they could not conceptualise it and try to identify mathematical relationships; and they chose not to use notes, only their bodies. At some stage the first author suggested that finding more curve points might help them understand what kind of curve it is.

Mary seemed puzzled throughout the activity, and after the break the teacher, who was engaged in another group during the particular investigation, asked what the students thought of it; the other student said "we were trying to find out another point" and Mary responded "we were struggling with it".

[196] The picture is included with the permission of the student.

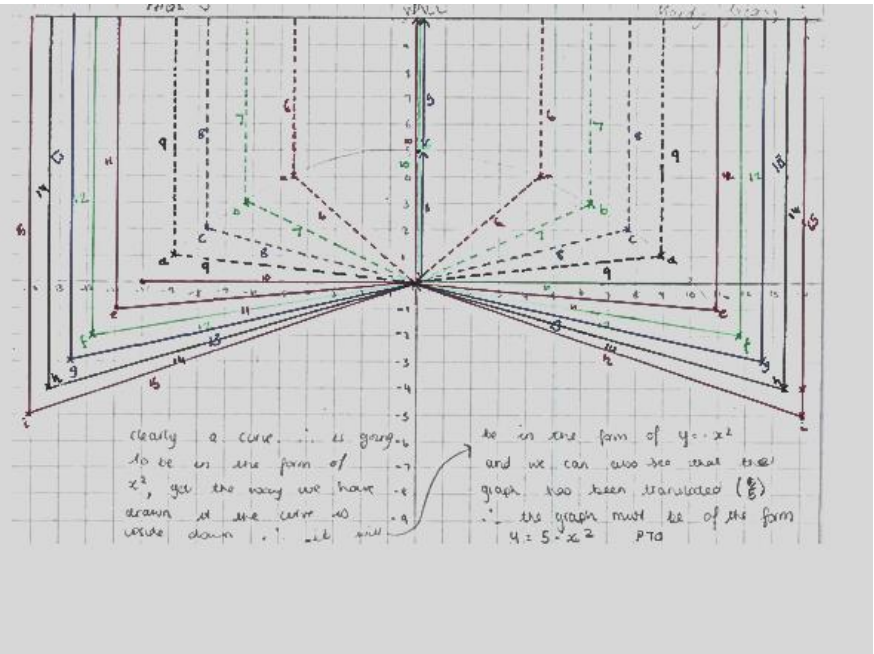

Figure 2: Mary's "little diagram", when it was completed. Later she added the formula for her final expression.

5.2 The diagrammatic second stage – The empirical 'bird's eye view' intuition

Mary started the second stage of her negotiation at home the same day with a diagram of the fixed point and the 'wall' (fig. 2 shows the same diagram when it was completed). Her diagram has its *intentional origins* in the portrayal of the classroom situation: "when I got home I just drew this little diagram … I think what I couldn't understand on the day [in the classroom] was that this wall went on forever!" (first interview). The importance of the diagrammatic stage invites us to look closely at how it happened; the original moment was the *empirical intuition* according to Husserl, which emerged and enabled the transition, as Mary described it in her interview:

> Once I could draw a diagram and look back [to the classroom experience], I look at it from *this* point of view; it's much easier than being in *that* point [i.e. the fixed point in the classroom], that looking, like, when standing here [she points to the fixed point on the diagram] *all you see is the wall!* [see figure 1] You don't see all this is going on as well. So once you can see from a *bird's eye view* it's easier to…

Her *intentionality* was fulfilled by intuiting the diagram as a 'bird's eye view' perception of the classroom. Her new perception appeared 'in one stroke', since it was not by induction that she realised that the sought curve could not be a straight line parallel to the wall, as she thought in the classroom; it just did not make sense anymore because she could now actually *see* it, from her 'bird's eye view' perception of the diagram. She also realised 'in one stroke' that her position in the classroom did not signify a straight line but a fixed point. So *immediacy* was an important feature of the transformation of the paper diagram to the classroom re-presentation and the dispelling of her classroom misconceptions. It is an example of the *embodiment of perception,* when the intuitive *moment* is also the *moment of transformation*.

In order to appreciate the empirical character of this intuition we need to look closer at its *visual* and *kinesthetic* features, and motility as basic intentionality (Merleau-Ponty, 2002a). As we focused on the intentional origins of the perceptual object (the classroom re-presentation through the diagram) we applied Husserl's theoretical ideas about lived-experience, in our attempt to explore how it came to life in a new sense. The diagram, more than representation of existing knowledge, was an embodied act that "constituted new relationships between the person doing the mathematics and the material world" (DeFreitas & Sinclair, 2012, p. 134). By "responding to the call" of the diagram (Merleau-Ponty, 2002a, pp. 161), kinesthetically and visually, her pre-reflective, *operative intentionality* was directed to the 'bird's eye view' of the classroom setting: It was an *embodied operation* that took place (not a positing thought), that 'I *can* identify the diagram as *such a perspective* of the classroom, here, on the piece of paper'.

We notice how the *operative intentionality* manifests itself through "what would be missing without it" (Merleau-Ponty, 2002b, p. 44, section 2.4): Mary acquired a particular *sense of space,* emanating from an *operation* of a plan perspective on the paper as a single object, which preceded any positing act; because I cannot posit anything before there is space for such a positing, and in particular, space that I can inhabit with my *living body.* It was only *due to* the new perception of the diagram by the operative intentionality that the reflective, positing

(intentionality of) *act* could appear, that 'I was standing there, at that point, and this line is the wall'. And it can serve as an example of how the *living body* is present in the perceptual field, as Husserl and Merleau-Ponty have predicted it (Husserl, 1970a, p. 106; also cf. Merleau-Ponty, 2002a, pp. 87-88).

It was not only Mary's diagram but also her classroom experience that acquired a new sense due to her bird's-eye-view intuition, since she could now "look at it from *this* point of view; it's much easier than being in *that* point, when standing all you see is the wall!"; the classroom's yet unexplored experience was re-presented on the piece of paper, ready for the student's mathematical interface to be applied.

5.3 The formal third stage of the investigation and the intuitions that took place

Mary's intention to mathematise the diagram was confirmed when she directly introduced the Cartesian coordinate system into it, thus entering the formal treatment of her investigation. She placed the fixed point at the origin and the wall became a straight line parallel to the x-axis (figure 2). Due to her 'bird's eye view' intuition every formal treatment that followed became equivalent to a classroom situation, thus imbuing her abstract ideas with an embodied empirical sense.

5.3.1 The empirical intuitions of the symmetrical points and the parallel lines

She easily found the first 3 points that belonged to the curve, namely the (10, 0), (–10, 0) and (0, 5). Then she *visually* intuited that the curve will be symmetrical to the y axis and while she was looking for more points, the first key idea in the formal stage of her enquiry appeared. She thought that if she would draw lines parallel to the x-axis and the wall in the area between the x axis and (0, 5), they would intersect the sought curve in 2 points (the grid paper has played a part in her empirical symmetry intuition, since she did not actually draw any lines parallel to the 'wall' and the x axis—see fig. 2):

> So I had this point, this point and this point… so then, I was like 'there must be another point; so what would happen if I'd change, if I moved this point so [that] it was on the line of y=4', yeah? (interview extract)

She recalls a *feeling of certainty* ("I was like 'there must be another point'") and there is already kinesthetic language involved ("what would happen if I'd change, if I moved this point"). Her empirical intuitive idea that the curve is symmetrical to the y-axis and that it cannot go any further than 5 towards the 'wall' made her parallel lines intuition possible. She saw these parallel lines as effective for a curve that was not there, apart from the 3 points that indicated the symmetry. And her feeling of certainty motivated her subsequent actions. The spatiality of her insight (the spatial reorganisation of the space between the x-axis and (0, 5)) and the actual sense of motion as she "*moved* this point" bear evidence of the *kinesthetic* features of her intuition.

Mary then "continued altering the y coordinates so that the distance from the wall changed" and "pulled out a triangle *in the hope* it will help us calculate what the x coordinate would be in order for the distance from the wall to the origin to remain equal". She involved the Pythagorean Theorem and her hope acquired formal ground by finding the first point (($\sqrt{20}$, 4)), and the whole process became a *technique* as she called it: "The technique seemed to be working so I decided to continue to use it".

The empirical intuition of the lines parallel to the wall with its kinesthetic features, based on the visual intuition that presented the sought curve as symmetrical, reorganised the space between the origin and (0, 5). Her intuitions constituted space within space (since the lines parallel to the wall and the x-axis represented for her their distances from the wall and the x-axis, respectively), and brought objects to the surface of consciousness (lines parallel to the wall and line segments perpendicular to the wall). Culturally mediated concepts (such as the coordinates, the grid paper and the Pythagorean Theorem) interplayed with kinesthetic and visual personal experience in order to constitute the new object (the parallel lines

as the way to fix the x coordinate) and the new tool/technique that realised[197] her new object.

5.3.2 The intuitions related to the x^2 curves

Mary had found the first 5 points of the curve ((0, 5), (10, 0), (-10, 0), ($\sqrt{20}$, 4), ($-\sqrt{20}$, 4)). According to Merleau-Ponty (2002a) attention "first of all presupposes a transformation of the mental field, a new way for consciousness to be present to its objects" (p. 33). The 5 points to which the student directed her attention to, became a *figure* on the *ground* of the paper and the other lines she had drawn. As she clarified in her second interview, "it was an aha-moment, I noticed that all of the points were going in the same fashion and thus there must be *a rule* as to what they were all following". A curve connecting these 5 points appeared in her consciousness. After she "realised that there was some pattern" she intuited that "this shape of graph", which she had just intuited out of the 5 points "is resemblant [sic] of the x squared graph". That is how her double intuition occurred: at first she noticed that there is "some pattern" between the 5 points and then the idea of "the x squared graph" appeared in her consciousness.

The extraction of a form out of the 5 points could be explicated as an *empirical* intuition according to Husserl and the result was the creation of an object, a shape that was brought to consciousness. The *immediacy* of intuition concerns the moment that the intentional act was fulfilled, the moment when sense was attributed to the intuitive *operative* intentional act, by the separation of the form out of the 5 points. The form of the 5 points has a pre-linguistic visual expression, as it transforms the real objects (the 5 points) to a new kind of object (a *profile* of the sought curve).

The second intuition associated the object that came out of the first one, with the x^2 curves. It bestowed the image of the first intuition with an abstract categorial feature, i.e. as belonging to the family of the x^2 curves, which is what

[197] The application of her new object in her technique is also perceived as a better *apprehension* of the newly-constituted object (cf. section 2.4).

makes this intuition an *abstract* (categorial) intuition. It is a moment in which we feel how the intuitive contact took place: the student's *operative* intentionality extracted the form out of the 5 points and her *intentionality of act* (to find "*a rule*") posited the form as an x^2 curve figure; It was "proper learning" (as she called it in the interview) that helped her to find an answer but it was only intuition that allowed it to appear in consciousness at the right moment.

5.3.3 Integration of processes – the constitution of the first formula

Mary objectified the set of arbitrarily chosen lines parallel to the wall and their corresponding perpendicular line segments (parallel to each other) and she used it as a *technique,* by applying it to the generation of another pair of curve points (($\sqrt{40}$, 3), ($-\sqrt{40}$, 3)). She then wondered if there was "a point that would be of equal distance to the wall and the origin … once the *moveable point* passed the x-axis". By application of her already objectified technique she overstepped her initial intuitive conviction that the curve is limited by the x axis…. It is a case that her formal treatment corrected the pitfall of her visual empirical intuition, showing that intuitions may be misleading but they are not necessarily end barriers; and it exemplifies how intuitive mathematics in action is legitimate mathematics.

At this point Mary employed instruction based arguments, without parting from her intuitive approach. She noticed that the x^2 curve is translated to the maximum point (0, 5), so she thought of the formula $y = 5 - x^2$. She tried the point (10, 0) in the formula but she got $-5 = -100$. And then she manipulated the false arithmetic relation into the correct formula $\left(y = 5 - \frac{x^2}{20}\right)$, by using the point (10, 0) and focusing on number 10.

Mary then tested the point ($\sqrt{20}$, 4) in her formula and she was convinced that the formula is correct. Two points ((10, 0) and ($\sqrt{20}$, 4)) were enough to constitute the sense of generality for her, which was something repeated later on (see the following section). The introduction of the formula by the student became an objectification of the coordinates of the points having the property of 10 units' equidistance from a straight line and a fixed point. Finally, her persistence in number

10 throughout the process became the preparatory step to her last intuition, which brought the general formula to the surface.

5.3.4 The intuition of essences related to the general formula

There is a particular kind of abstract intuition, which is a *categorial* intuition that Husserl calls *intuition of essences* and it is characterised by apodictic evidence (Lohmar in Haritmo, 2010, p. 74). The example of Mary's last intuition will allow us to get a glimpse at the main features of this kind of intuition, which holds particular significance for generalisations and mathematical proofs.

Although Mary had answered the particular mathematical enquiry, her intuition pushed her investigation further, by objectifying the whole process into a *new tool* in order to *extend* the task. The distance between the wall and the fixed point remained her intended number as she repeated the whole process and found the formula of the curve when the distance between the fixed point and the wall is 6 units instead of 10($y = 3 - \frac{x^2}{12}$). As she explained in the interview she had already seen the essential relations between 10 and 5, 20 in the first formula before she figured out the second formula. But she became *increasingly* certain concerning the part of number 10 in the first formula, as she applied identical processes to the second formula. And she acquired the formula $y = \frac{a}{2} - \frac{x^2}{2a}$ of all the curves with the same property, by replacing 10 by a variable in her initial formula.

The particular qualities of the *intuitions of essences* surface here, such as the three 'moments' that conjoin, in the following sequence:

- The first formula is the *starting example* (Lohmar in Haritmo, 2010, p. 78), the *object of experience* (ibid., p. 83) and it became *the leading example*, already incorporating essential embodied elements as the previous analysis has shown.
- *The generality that the leading example gains through the modeling of the whole process* in another, intended example (the second curve). The matching of the first formula to the second one (and the correspondence between 10 and 6) was decisive for the process of its generalisation.

- *The "synthesis of coincidence"* (ibid. p. 79), through *the recognition of number 10 as such,* stripped of its particular originations and seen as a 'variable number'. This operation is instantiated in the leading example, by pinning 10 down to the mathematical symbolic frame and transforming it to a variable. That is how number 10 was recognized as essential for the expansion of the formula and finally acquired a new quality.

The 3rd moment is the conclusion and the core of the intuition, since all steps were realised in a stroke, in the recapitulating act of coincidence (the third 'moment'), typical of the intuitive *immediacy*. Mary sidestepped the various connotations of number 10 (sensory, algebraic, geometric) and identified it as a number—as 'any' number. What made this possible is what Lohmar calls the interpretation of the "special synthesis of coincidence" (Lohmar, 2010, pp. 79, 80; also cf. Husserl, 1973, Section 1), based on the singular example of the first formula and the generality it acquired from its reflection on the other formula and curve through number 10. The '*synthesis of coincidence'* concerns the *qualitative transformation* - through a novel *thematic identification* - of the particular, out of the awareness of the generality. The synthesis of coincidence occurs between the subdividing acts - the two formulas. It is the apprehension of the coincidence as the identity of the general feature that becomes the *essence*, expressed as the general formula - the intended object of a *new category*.

5.4 Coexistence of abstract and empirical intuitions as a sign of deep understanding – the curve as a "moveable point".

The phenomenological analysis allows us to see the linkages between the empirical and the abstract (categorial) intuitions, and the transition from the pre-reflective to the reflective field of experience.

The student perceived the sought curve as a "moveable point", which is an expression she repeatedly used for the curve in her coursework. Her sensorimotor expressions ("moveable") (see the quotation in 5.3.3) are signs of how she explored the relationship between the locus of the parabola points and the functional one, as she vividly described it when she had accomplished the task:

> Like for that I am positive that I can say that if this is true and if this part was *a,* you could stand on anywhere on the *line* $y = \frac{a}{2} - \frac{x^2}{2a}$ and you would be equidistant from the wall and a fixed point. (first interview extract)

So the formula could be rolled back to its empirical origins, integrated in a perception that permeated the student's investigation: it's always points, either points to be traced with lines parallel to the wall, points to judge if the formula is the correct one, to manifest the generality achieved through the concrete particular case, and so forth.

Finally, what is remarkable and pertinent to this research is that due to a sequence of successful intuitions she constructed formulas that she only later knew were called parabolas and a general formula, in a bottom-up process. It was this lack of names and ready-made information often confused with knowledge that enabled her to realise the parabolas, through their equidistance property.

6. Discussion

The research supports the claim that phenomenology, being a philosophy of the 'immediate experience' can give a convincing account of the significance of intuition in the objectification process.

The two-level structure of intuitions (empirical/abstract), as it was introduced in sections 2.4-2.5 and analysed in the data (e.g. 5.3.2) corresponds to ideas expressed in non-phenomenological literature. For example, Fischbein points at primary intuitions as the cognitive beliefs that develop independently of any systematic instruction as an effect of their personal experience, which corresponds to the Husserlian empirical intuition. And Husserl's abstract intuition is similar to Fischbein's secondary intuition (1994, p. 68) that has no natural roots, and the Piagetian operational intuition (ibid., ch. 4) that comes from special action towards objects of the world, rather than mere perception. By employing the Husserlian theory, the similarities and differences between the two levels of intuitions were clearly shown in the data analysis, without any appeal to idealist or biologistic concepts, such as the concept of the adaptation mechanism (ibid. p. 59), but by taking into account their position and functioning in the objectification process

But what is principally new in Husserl's approach is intuitions as objectifying acts, which separate a new object from unstructured, unthematised raw material. The bird's eye view intuition brought up the diagram as a new object from unstructured sensory-motor classroom and home experience, and the general formula appeared as identification between elements of particular formulas, which became essential only due to the fulfillment of the intentionality of the general formula. Here we deal with the learner's lived experiences as separations of lived profiles of the learner's reality, not as already categorised Kantian perceptions.

Empirical intuitions are mediators of our lived-life, they are accumulations of experience of our embodied consciousness and they bring new objects that we *come to learn as we use* rather than construct. They are embodied, pre-objective operations, *followed* by action. Empirical intuitions concern the transformation of the sensory, unreflected, raw data to forms, to empirical objects, which become the ground, the subsoil of the abstract ones; (e.g. the 'bird's eye view' intuition that transformed Mary's classroom experience into the diagram and became the ground for her further abstract intuitions (section 5.2)).

Abstract intuitions bring objects of a new kind with them from 'ordinary' singularities, such the figure that emerged from the 5 points on the piece of paper and was linked to the x^2 curves (section 5.3.2), or the numbers—distances on her diagram—that were intuitively targeted and evolved to a technique and then to a variable, in Mary's general formula (section 5.3.4). Abstract intuitions were shown:

- to follow similar patterns of emergence to corresponding empirical intuitions: e.g. the figure that emerged from the 5 points appeared on the ground of the grid lines of the paper and other lines she had drawn, in the same way as the x^2 curve graph emerged on the ground of other graphs (section 5.3.2)
- to work on the products of empirical intuitions: e.g. Mary has recognised a pattern in the 5 points, which became a basis for the x^2-curve abstract intuition, or e.g. number 10 was intuited as an important one and then became a basis for abstract intuition through its involvement in the formula production

- and to bring new sorts of objects to the surface (e.g. the general formula in section 5.2.4), by focusing on invariable features of previously constituted objects (the two formulas), in the context of their cultural (mathematical) horizons.

We consider Mary's last intuition towards the formal expression of the parabola (section 5.2.4) as a particular kind of abstract intuition, the *intuition of essences*. It illuminates the essential properties of two previously objectified forms; but the interaction between forms keeps the starting example as the principle knot of reference. In this case the general formula appeared as the essential property of two particular formulas and the first of them stood as a reference to Mary's embodied experience. Thus the synthesising intuitions of coincidence (essences) become the embodied, personal and at the same time cultural agents of learning and teaching experience. The intuition of essences revealed not just an abstract knowledge of formula, but rather a node in the system of layers of objectified experience grounded in the *living-body*.

From one hand it is the *living body,* the body-subject that lies at the core of intuitions, from another hand it is the intuitions, the starting points for the discursive nature of knowledge, since they link the pre-objective raw material (hyle) of lived-experience to the reflective conscious treatment of cultural (mathematical) objects. During the objectification process objects emerge once the students become conscious of the effective implications of their intuitions and then the objects evolve through their further involvement in the students' activity towards the next objectifying acts, in order to arrive to the final cultural form. It is in this sense that intuitions of essences, as well as all abstract intuitions are the end points of the learner's analysis and "by the same token the starting-points of our epistemological and constitutive processes" (Hintikka, 2003, p. 180).

We can see here that one of the novelties of Husserl is that he gives a legitimate answer to Meno's paradox of the acquisition of knowledge, through the subjective cognitive praxis, which is transcending its cultural mediation at the same moment that it is culturally transcended by pre-existing tradition and similar constitutive practices. Being deliberately bracketed by phenomenological attitude

cultural conditions took significant part in the investigated process of objectification: it was the cultural practice of modelling that allowed Mary to draw the diagram; it was the ruler as a cultural tool that allowed her to depict a straight line on the paper; it was the specially structured grid paper that facilitated the application of the coordinate system and her visual intuitions, and so forth. But it was only personal lived experience that let objectification happen, that made possible the coincidence of intuitive synthesis in Mary's last intuition. Thereby we suppose that Husserl, who has radically rethought Kantian and Cartesian dichotomies, could be considered as complementary to current cultural-historical approaches, from the point of view of the subject-in-the-world.

I exemplified how the lived experiences genetically break down the dualism of "outer" and "inner" world, since they are "appearances-of-something… of existing things presenting themselves" (Held, in Welton, 2003, p. 28). *"[T]hey are the In-between, that which originally opens the dimension of intentional appearance within which consciousness and the world have already met—before any subject-object rift"* (ibid.). It is in the beginning, in the origins of objectification where the genetic issues of the constitution of temporality (Husserl, 1991, 2001) and of the "unity of coincidence corresponding to the unity of sense" (2001, p. 44; cf. 1970b, 1973) arise. There—in the origins of objectification—are structures that persist in personal cognitive experience, namely the synthesis of coincidence, the recognition of the generality from the mirroring of a starting example and the double operation of categorial intuition. This double operation of intuition (section 5.3.2) starts from an empirical particular that is constituted—not merely found yet not constructed—and then used for the categorial association in order to introduce the presented curve figure to the x^2 family of curves. Making association possible is a *poetic moment* (Radford, 2010, p. 6)*,* which Husserl's theory of experience makes visible, where a student is engaged with her living body that manifests itself in its actions, starting with its intuitive body-subject-in-the-world constitutions.

7. Implications for teaching and learning

The phenomenological perspective of our research approaches an understanding of cognition where intuitions and their objects are seen as essential sources of meaning during the learning and teaching practices. Adopting this position one becomes attentive to the individual learning history of objects, to the sequence of the learner's intuitive moments that make possible the evolution of objectification from raw material towards formal mathematical knowledge. The history of the student's concept of the x^2 curve in this study appeared as full of small discoveries and misleading assumptions that were overcame by her subsequent intuitive acts.[198] In this sense, the individual learning history may be more instructive than the history of a concept. We support the teacher's desire to learn from the students' particular ways of learning, to communicate these individual perceptual approaches with the other students, and to reflect on her own. We offer our analysis of the student's unique learning experience as a leading example for a teacher who would venture to bring some implication of phenomenological attitude into teaching practice.

The phenomenological approach of the research suggests that unfolding the multiple stratifications of the subjective perspective of the learner's lived-world is equivalent to letting actual engagement of the learner emerge, and the learning praxis to take place and be recognised as such. Focusing on the students' intuitions and objectification processes is an approach that leads to the core of their learning practices, since intuitions, as particular expressions of perception, are connecting the hyletic data, i.e. the "unstructured raw material" with "our structuring and form-giving mental activities" (Hintikka, 1995, p. 98). Nurturing the students' intuitive processes, facilitating them with corresponding intentional teaching adjustments, and aiming for the "intuitive starting point" (van Hiele, p. 122) are suggested forms of action.

The research shows how in fact more delicate is the teacher's task, since it requires letting knowledge emerge and acknowledging its rich grounding in the learners' intuitions, rather than confining it to her own perceptions by offering ready-

[198] She 'saw' that the sought curve is not a straight line from her bird's-eye view (section 5.2) and she removed her previously intuited x-axis barrier by applying her just objectified 'technique' (section 5.3.3).

made answers, and utterly reducing it to "the procurement of useful information" (Heidegger, 2004, p. 15):

> Teaching is even more difficult than learning. We know that; but we rarely think about it. And why is teaching more difficult than learning? Not because the teacher must have a larger store of information, and have it always ready. Teaching is more difficult than learning because what teaching calls for is this: to let learn. The real teacher, in fact, lets nothing else be learned than—learning. (p. 15)

The research can thus contribute to "the emergence of a complex collective whose interactions and products are mathematical", in service of diversity that is assumed and flexible rather than legislated (Davis & Simmt, 2003, p. 149).

8. Conclusions

We regard the results of our research as contributory for both theoretical understanding of learning and practical application in learning and teaching.

The phenomenological perspective advocated by this study opens a possibility to disclose the mystery of "[h]ow will you set about looking for that thing, the nature of which is totally unknown to you?" (Merleau-Ponty, 2002, p. 431): at the very beginning of a new knowledge it is the living body and the lived world who meet beyond the object-subject dichotomy, in order to shape objects by intuitive acts. It is the origin of the objectification process when intuitions bring objects on the surface of consciousness, shaped exactly in accordance with the operative intentionalities of the student. Intuitions reveal the objects that emerged in sensory-motor behaviour, as we have shown by the analysis of the student's visual and kinesthetic experiences. Despite the cultural practices that provoke and the cultural tools that form this behavior it is exactly the embodied consciousness of a transcendental subject—a student in our case—that was populated by objects. The intuitive constitution of the first objects appears without any predisposed categorical structure in the outside world, which is constituted together with the objects. This is the very subtle and specific role of intuition that phenomenological analysis reveals and that escapes being perceived from Kantian-inspired methodologies.

As we have shown, Mary's process of objectification was enriched by empirical intuitions that transformed hyletic material into empirical objects, and by abstract intuitions that transformed her lived world (already inhabited by empirical objects) into abstract objects. The contention of this research is that teaching diversity already arises from the awareness of intuitions and their embodied ground from which the variety of ideas, strategies and approaches emanate for the learner. By adopting a phenomenological point of view the teacher may become perceptive to the students' intuitions and attentive to their conceptions, which will in turn facilitate the objectifications of their understandings. Teachers' orientation towards the understanding of the student's living body experience is a step towards an inclusive, student-centred teaching frame. It is a suggestion towards a critically realistic interrogation of teaching and learning norms that ignore the student and, eventually, the mathematics (Polya, 1988; Lakatos, 1978).

The inclusion of pre-objective and pre-scientific material in teaching and research frames, allows the knowledge of the body and the embodied consciousness to contact the mathematical significative domain rather than be "silenced by the sayable and the linguistic" (deFreitas & Sinclair, 2012, p. 149). As Merleau-Ponty (2002) summarises it, "all knowledge takes its place within the horizons opened up by perception". In this process intuition is the essential objectifying act.

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Appendix B. Ivan's and Carry's interviews during the New York cop session

Paragraph 1. Ivan's short interview, 10/2/2011, 12:11am

The interview was taken at the beginning of the second part of the *New York cop* session. I was participating in Ivan's group and, in this sense, Isaac's interview is also a cross–checking process. As I had already familiarised with him and his work in the group I started the interview without any introduction; it was the 5th out of the 7 interviews I took that day and the interview was already running out of time, although the time that I had available for the interviews was not strictly set. I am using conversational analysis (CA), after Penakülä (Handbook of Qualitative Research, ch. 34) for the transcription of this interview, in order to transfer the narrative rhythm:

R: Hello Isaac, [I place a chair next to mine], sit there please. [Isaac sits and I sit] Alright, just tell me how you started with it, and how you got to the point that you are now?

I: Eh:[199], I started off deciding to look out squares, rather than (.)[200] increasing the columns.[201]

R: What kind of squares?[202]

I: Eh:, equal squares,[203] 5 by 5.

R: Ah, all kinds ((sizes))[204] of squares!

I: Yeah, yeah.

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[199] : Prolongation of sound.

[200] (.) Pause of less than 0.2 seconds.

[201] Isaac's interview took place after Caroline's one—which was the 1st interview—and he used the 'columns' terminology before I said anything about the activity.

[202] I leave no gaps in Isaac's narrative, in order to maintain the fast narrative rhythm. My question does not sound as having any sense, but it is prompting Isaac to describe what he did with the squares.

[203] The last word ending with a questioning intonation, to be interpreted as "is that what you are asking?"

[204] (()) Transcriber's comments that explain what was just said or happened.

R: 2 by 2, 3 by 3,

I: Yeah.

R: 4 by 4, 5 by 5 and (.) and so on....[205]

I: Yeah!

R: [I make a sound prompting Isaac to keep talking about it]

I: Then ehm, looking at how: (1.9)[206] the patterns form, (.) to do with, (1.7) this size ((he is pointing to one of the squares in his notes during the activity)) such as eh… (1.5) I'm thinking about where would be the most sensible place to put a point ((i.e. a cop)); (1.6) so that you get the optimum solution.

R: [I make a sound prompting Isaac to keep on talking about it]

I: Eh:.

R: [207] And trying to find patterns I guess.

I: Yeah! I decided that it would be most sensible to work from the (.) *out*side.[208] Once you've done one ()[209] of the outside, deciding which points from the *in*side, you need to cover (0.9) next.

R: [I make a sound prompting Isaac to keep talking about it][210]

I: To cover all the squares.

R: [I make a sound prompting Isaac to keep talking about it]

I: As well as you are working away to the () (1.0) crossing up bits.

R: [sound prompting Isaac to keep on talking about it][211] And eh, what did you get with that? I mean did you find something out o:r…

I: Well: I fou:nd that (.) all of the patterns are sy*mme*trical, along a diagonal (0.5) eh, line

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[205] I am introducing a systematic approach of squares, shifting the conversation to what I had already observed and to the mathematical potential of his treatment, as I saw it.

[206] (1.9) Silence measured in seconds.

[207] Isaac does not respond, and I try to maintain the fast narrative rhythm, as in footnote 5.

[208] *Out*side Emphasis.

[209] () Inaudible word.

[210] Consult previous footnotes concerning the continuance of the rhythm.

[211] Isaac does not respond, and I try to maintain the fast narrative rhythm, as it is described in footnote 200.

R: [I make a sound prompting Isaac to keep on talking about it]

I: Eh, (2.7) [checking his notes] *e:ven* symmetry

R: [I make a sound prompting Isaac to keep on talking about it][212] Any:, any formula on that or…

I: Eh, not a formula *yet.*

R: [I make a sound prompting Isaac to keep on talking about it][213] Have you been engaged with eh, rectangles?

I: Eh, no not yet.

R: [I make a sound prompting Isaac to keep on talking about it]

I: I (.) decided to focus on squares (.) first.

R: [I make a sound prompting Isaac to keep on talking about it][214] So you found patterns fo:r squares; is there any regularity there? (2:03)

I: (1.0) Eh: (.) Other than the diagonal symmetry (.), and the number of (0.8) columns of most of them so far. (1.0) There's a (.) 5 (.) diagonal (.) lines (.) from which they go along (1.5) or most of them.[215]

R: You mean that the number of diagonals (.) is equal to the number of eh: rows or columns?

I: Eh, no, just (.) 5. Like this eh, 5 diagonals, like 5 diagonals, like 5 diagonals, like 5

R: So there are 5 diagonals in each case? Not in each case but, after th… three

I: Well, after three, and I'm assuming I'm not wrong to say that...

R: To *4.*

I: Yes, well, after 3, because 4 is after 3!

R: Yeah, yeah.

I: [laugh]

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[212] Isaac does not respond, and I try to maintain the fast narrative rhythm, as it is described in footnote 200.

[213] Isaac does not respond, and I try to maintain the fast narrative rhythm, as it is described in footnote 200.

[214] Isaac does not respond, and I try to maintain the fast narrative rhythm, as it is described in footnote 200.

[215] He refers to the diagonals of the cops' placement in the 3 by 3, 4 by 4 and 5 by 5 precincts.

R: Yes [laugh]

I: I'm assuming that it will go on to s… six or (0.8) so on. (2.1) I don't know really

R: So is that all (.) you've done (.) so far?

I: Yeah. I'm collecting data at the moment if I can make a (.) generalisation.

R: Alright. Ok?

I: [approving nod]

R: Thanks. (3:14)

Paragraph 2. Carry's short interview, 10/2/2011, 12:11am

Researcher: Please tell me how you started and where you've come so far and which way did you…

Carry: Ok, the way we begun, eh, the teacher just said to think about how the placement of the cops would change as you expand the precincts, and those different ways to expand it; so first I had to look at the expanding at adding on a column, eh, to a block each time, eh, making… you know squaring, the blocks, and putting, eh, like a path of blocks around each time, so it's expanding.

R: Square or?

C: This would be the first one and then, you put this round and this one [the student holds the notes she kept during the task and she refers to these notes]

R: Like the 'garden-path' [activity].[216]

C: Like the garden-path, yes. But I haven't gone into that one [laugh]. So I started off with the easiest one which was adding that column of block[s] on each time.

R: Keeping the [number of] rows the same or…

C: Keeping the [number of] rows the same, [that is] 2, 2 [rows]…

R: 2 by something

[216] The way she described her strategy reminded me another activity, where the 'garden' was consisted of squares and the 'path' was consisted of squares surrounding the 'garden'.

C: Yeah, 2 by something; but then if it was the other way around, increasing the rows it would be…

R: It would be the same.

C: It would be the same, yes [laugh]. (1:15) So, eh, I did by the first ones 2 by 2 and then 3 by 2 and then 4 by 2,[217] and, eh, I was seeing, from my optimum placements that the number of cops was **one more** than the… the row length?[218] [her questioning intonation at the end of her phrase expressed uncertainty about the expression 'row length' that she used][219]

R: Yes.

C: For 2 it was 3 cops

R: The amount of columns or rows?[220]

C: Yeah, the amount of columns, sorry yeah.

R: Yeah.

C: And then I put it into a table and… so, and then I started looking at it and **why** it was like this; and then, for my reasoning it was that because one cop is needed to… eh…

R: To watch.

C: To watch each... I was calling them just the *'vertical streets',* because that's how they are in my paper; and all the streets in between, so you would need, eh, one to go, to cover that, so…

R: Yeah.

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[217] The first number refers to the columns and the second one to the rows of each precinct.

[218] It seems like she has not yet found a 'fixed' name for her empirical intuition, concerning the part that the expansion of columns plays in relation to the cops' expansion, when the rows are kept invariable. Giving a name to the columns' expansion amounts to constituting a profile of her experience; hence endowing a discursive sense to the pre-objective materialisation of the qualities that she has already perceived.

[219] Although she is thinking 'vertically', by counting the amount of cops in each column and adding more columns, she is expressing 'horizontally' when she says "the row length".

[220] I am instinctively trying to 'fix' the conversation using the 'rows'–'columns' terminology, rather than 'row length'.

C: So that would be the… all of the columns plus the one in the end, or plus one; depending the way you look at it. Because it's only 2 rows… yeah! It works. Like that, so if it was getting increasing this way or this way, then it changes of course, but because it stays on 2 rows then you can do… you can fit in a thin…

R: So your basic idea is, is, how would you describe it if you like?

C: Eh, as you add a column at each time, you will need one cop more than that column, to cover all the streets. That was the conclusion I came to.

R: And that's when you got 2 rows.

C: When… yeah, it's like 2 rows and…

R: And no matter how many columns.

C: Yeah.

R: Have you experimented with any other pattern to see if there's any regularity there? Say, 3 rows or squares or whatever?

C: Eh, well, I could look at 3 rows but then I decided to look at square precincts, 2 by 2, 3 by 3, 4 by 4.

R: Did you find any regularity there, did you discover anything?

C: So I just started experimenting a bit and placing them and… comparing my results with [those of] the people on my table [the other members in her group], and we sort of agreed on the optimum placements [of the cops]… so **then** I decided to try and find more of a **pattern,**[221] because it wasn't really **that clear** with this one; you know, what **the pattern** was, so… and we[222] began trying to place the, eh… the cops in the same sort of way each time but [when] it gets bigger it gets really difficult! To do, eh… so, there's a lot [unintelligible words] the cops around the outside, we realised that eh… you can always work out the minimum number of cops you'll need on the outside because of how far they can see. If it's 2 by 2, it's just 2 [cops] on the outside.

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[221] She shifted to checking regularities in the **image** of the cops' distribution in the squared precincts.
[222] Her tense turns to second personal.

R: If it's 3 by 3 you'll need 4 [cops on the outside].[223]

C: 3 or 4, 4. Between 5 and 8 you need 6. And you can carry on working this up.

R: So you concentrated on the outside cops.

C: Yeah! So I separated it 'cause…

R: And then you would check the inside cops?

C: And I would start looking at the inside ones or trying to find a pattern to see how they change each time, so… So I started like this and then 2, then, and so it was looking, the first three, it's, it's doubling [laugh]. Eh… But then, this one is the next one, so it's still following a pattern, it's from here to here, with the 6 now in the inside.[224] And this is the next one, so now there is…

R: So there is some pattern.

C: There is a pattern, yeah, and it's getting difficult now, you can see [laugh]. (6:03) Crossing out more as they get bigger.

R: You are using different colours for the outside cops and the inside cops?

C: Yes, I think it's easier to get to visualise the patterns; and that's where I'm up to really; and, yeah, I'm just trying to find a pattern so that I can see how that… how that changes as it increases, and so that I will find something general

R: So is it how you understand generalisation in this activity?

C: Yeah! Yeah, and this is the way that I'm choosing to look at it. I don't know why [laugh], because it's just the way itself, it's so, I see it, I just saw it as two separate things, the outside, the cops at the outside…

R: So you are trying to manage with the shape and then come out with a formula.

C: Yeah, yeah.

R: To manage the shape, whatever the shape might be and then you might come out with a…

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[223] I'm just saying what I see in her notes, in order to keep the conversation flowing.
[224] She talks about the 6 by 6 precinct, where there are 6 cops inside.

C: To try and find a common way of increasing… how the cop pattern changes as it increases.

R: Could you repeat [what you said, since] there is a conversation at the corridor?[225]

C: I'm trying to find… what I've been trying to find is not a formula, I'm trying to find a common way of increasing the cops each time, and seeing from that pattern something that matters.

R: Yes…

C: I guess; it's not really focus on formulas at all [laugh], [I'm] not bothered with that. Yeah, but it's interesting, I like it (7:37).

R: That's your personal way of doing it, that's interesting too.

C: And that's as far as I've got now. So I will continue with looking for these patterns; but there are just so many ways of expanding it and continuing with the problems, that you just **have to choose something!** [laugh] (8:00)

R: You don't feel that you've managed it yet but, you're moving on…

C: I will, it has to be taken into account [as] you're saying well, what if I you had 3 rows. You'd begin with 3 rows and then you're increasing the columns. Well, I'm sure you know I could find eh… a common pattern for that as well, but I haven't looked at that, yet. But I could do; there are just lots of things, but you could do, so, you **have to** narrow it down to manage it. But I might go back to that.

R: Ok, that's fine.

C: Ok.

R: Thank you. (8:42)

[225] The conversation takes place at a corridor, two people appeared and started to talk to each other and I did not hear what she just said.

Paragraph 3. Ivan's lengthy interview, 7/4/2011

One of the reasons for this interview was the disappearance of the recording of his presentation from my digital recorder. My purpose was to test my assumptions during his presentation, to explore his conception of his own presentation as well as the other students' presentations, his perception of mathematics and mathematical objects, and his conception of what is a 'good' teacher.

Ivan was writing some notes concerning his activity while his was waiting for me.

Researcher: You are doing something about your activity, aren't you?

Ivan: Yes, is anything you…

> [He sounded a bit 'tight' and I interrupted him by telling him what I really thought about his activity, thinking that it would encourage him:[226]]

R: I found it **very** interesting

He immediately loosened his way of sitting and he smiled.

R: It had various objectives…

I: Yeah

R: and I found it extremely interesting

I: Yeah, thanks

R: I'd love you to give me some clues about it, how you've picked it up and what you were aiming at when you chose it

I: I first came across when I was reading a book, which I got for Christmas of my mum, called "Alex's adventures in number land"; and I thought it seemed like quite a nice little sequence, **pattern type thing**.[227] And I've seeing Sierpinski's triangle before, so I thought "I'll have a look at that"; a square one would do, it was the

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[226] My aim was to conduct the interview in a 'casual' manner, with Ivan talking most of the time, and minimising my intervention to the clarification and the broadening of the topics of the conversation (semi-structured interview).

[227] My emphasis; Isaac's attention to the activity that he decided to present seems to have been initially drawn because of its pattern type aspect—since Sierpinski's carpet is actually a fractal. It was "later on" that he realised its potential for generalisation in 3D shapes—as he admits in his following comments—although he was probably aware of the Sierpinski triangle—i.e. the expansion of Sierpinski's idea in the 2D space.

thing mentioned because I knew that later on you could do in a 3D shape. So I started looking at the two dimensional Sierpinski's carpet. The main thing that I wanted to look at is area rather than vertex or edges because it would be quite simple to account.[228] So I looked at what the area of the first one would be, which would just be x^2. Then what would happen when we remove the middle [square] of the three by three grid, so get the x^2- $\frac{x^2}{9}$, which got me $\frac{8}{9}x^2$; then the smaller one, which ended up being $\frac{64}{81}x^2$. It wasn't until the one after that, the $\frac{512}{729}$, that I realised that it was $\frac{8}{9}$ to the power of n-1, times x^2 for the area. So, once I worked that out I thought "okay, what happens with the cube?" and, "should I look out with the cube?". So, first thing I did look out, it was volume, because it looked the logical step area for 2D shape to 3D shape

R: Yes, from area to volume

I: Yes, and that was a bit more complicated because you are imagining spaces which can't see as well; because you can't draw a three dimensional cube with all the lines 'cause you just confuse yourself immensely.

> At this point I was tempted to test (without having any intension to insist) Isaac's attitude towards mathematics as an abstract field, when 'reality' or a 'concrete' model of it can no longer provide any help. I said the following, trying to make it sound more like a question than a statement:

R: So you have to use abstract tools in order to manage it, yeah?

I: Yes

R: And do it with just empirical methods [like] "I can see this there, I can see that there"

I: Yeah, a [an unclear word] of imagination as well

[228] It seems important for Isaac that the task ought to be a demanding one.

I did not spot any particular tendency to explore this dimension so I decided to change the subject. I decided to explore his motivation when choosing the particular task.

R: So it really **excited you** to check this activity and that was the primary reason because you liked it

I: Yeah!

R: Why would you think that it is useful to anybody there?

I: I didn't think about anyone else, I just wanted to do it, to see what happens[229] (3:12)

R: Yeah, although I saw that you cared quite a bit so that **everybody** would understand it

I: Yeah

R: You've been around, you were walking around, you were helping everybody… in fact, to be honest I thought that you gave a bit more information than was necessary

I: **Yeah, I kind of wanted people to get to the answer because that's… I wanted them to understand rather than to just be left in darkness trying to find their own way out**

R: Did you worry when I was choosing a different way of doing it –I was going through series,[230] I don't know if you realised that

I: It kind of worried me when you started writing two to the power of three and three squared and…[231] I was thinking "you don't need to do that! Just keep it simple". It's the best way rather than over-complicating it

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[229] It seems that the gratification that he is personally getting from the involvement with the task is Isaac's primary motive.

[230] That is infinite sums of sequential numbers ($\sum_{n=1}^{\infty} x_n$).

[231] He refers to the expression I wrote for the case n=4: $x^2 - \left(\frac{x}{3}\right)^2 - 2^3\left(\frac{x}{3^2}\right)^2 - (2^3)^2\left(\frac{x}{(3^2)^2}\right)^2$, since he wanted us to end up with the formula $\left(\frac{8}{9}\right)^3 x^2$ (for the case n=4).

R: I was trying to formulate something like this: $x_1 + x_2 + x_3 + x_4 + \cdots$

I: A geometric sequence

R: Also a sum: $\sum_{n=1}^{\infty} x_n$

I: A sum of all… to infinity

R: And you [may] stop at any stage; if you do three stages that's it, if you do ten [stages] that's it. And then you can see that it converges through the general algebraic formula that **you** have found

> [At this point I realised that I got a bit carried away with the mathematical aspect of the approach that I had during his activity, in my attempt to test how he reacts towards a different mathematical approach (one that he had not thought of), and what was his strategy for alternative approaches that might occur during his activity.]

I: Yeah

R: It gives a different dimension to the activity than the one that you already thought; and I thought "he is worried because I'm going through another 'path'…

I: Yeah!

R: …and I don't get to what **he** has thought of it"

I: Yeah!

> [I remained silent waiting for his reaction]

I: Well, I suppose there is probably a path to go to get it and I just wanted to guide people on my path rather than having them exploring their own routes; and yours just seemed to confuse me, so I tried to keep you on a 'narrow road', not allow you to 'go into the woods'

R: How did Lewis react to it? He was pretty fast, wasn't he?[232]

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[232] My sense during the activity was that Lewis has been at least as fast as I was, and probably even faster (see the file *Reflection on Isaac's presentation*); one of the reasons asking this question to Isaac was to test this assumption.

I: [He was] Pretty quick, not sure how quick he was because I didn't have to deal with him much, because he didn't need help.[233] And I noticed a few people [that] just copied of each other in the end.

R: Copied?

I: Tim copied Lewis. Because Tim didn't understand anything to do with it and then he had the answer right up the end when Lewis had it, so…

R: What about David?

I: Dave…

R: Did he copy Tim, who copied Lewis?

I: I don't think that Dave copied! I think that Dave just left confused, as Dave usually is left confused

R: There were other people who just didn't cope with it, did you feel that?

I: The table[234] that was here, which was Barbara, Joanna, Kirsty and Sindy, they just sort of lost interest and started talking about other stuff, and I couldn't control them; and every time I went over and tried to help them with some of the even simple stuff, just finding the area of one of the cubes, which have removed, they got confused over that. So that was a problem because I wasn't expecting them to get confused

R: That [at least] one of them would go through, and so the other ones would go along with it. Or Diana, she was pretty confused I think[235]

I: Diana was pretty confused but she slowly grasped what I was trying to get at with how x over three becomes x squared over nine, when you [an unclear word] it together, and how you work out the lengths of each side to [an unclear word]. At one point she started eh… what she did? … she was doing something really weird

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[233] Isaac confirms here my assumption that he was providing feedback to all those who would care for it.

[234] He is talking about a particular group of his peers when he refers to "The table".

[235] Diana was sitting next to me and I noticed that she was expressing difficulties managing the algebraic aspect of the activity (see the file *Reflection on Isaac's presentation*).

with breaking each… breaking it down into its ninths again, then removing it from that… and get back up

R: Separate pieces

I: Yeah, so she looked at that just bit, work out what that was and to get the rest of it rather than just removing it bit by bit…[236]

R: Typical by Diana, she is 'breaking' things as…

I: Yeah!

R: …to the easiest kind that you can think of; then she sums up, she…

I: Yeah!

R: …she insists in getting there and she doesn't give up

I: But the problem was that when she was doing that she was making mistakes which…

R: But you were there! I noticed you had an eye on everything…

I: Yeah!

R: …you made sure that nobody would…

I: Yeah!

R: …lead astray "in the woods", you know…

I: Yeah!

R: …going there around

I: I was glad Caroline got to an answer, as well; that, that made me happy. She managed almost independently, and how she got all these results; she was one

[236] It appears that Diana noticed the fractal quality of Ivan's activity (see footnote 4 in *Reflection on Isaac's presentation*) and she was working with a part of the shape in the same way that she would be working with the original shape. It was Diana's particular way of trying to make the activity manageable by 'breaking' it into smaller manageable pieces, and it is quite interesting how Ivan unfolded Diana's different construction of the activity, different in the sense that Isaac was not expecting such a treatment ("she was doing something really weird").

small step away, from getting an answer, and she couldn't see that, she was so… (8:16)

R: Did you give it to her?

I: I didn't give it to her in the end, at all, to how she may come about it; I said to look at the powers and to do with the iterate issues rather than just looking at the numbers itself. So…

R: Then she got it

I: Yeah, then she got [it] a few minutes later. So, I was kind of happy about that

R: So **it was** important for you that people would be engaged and come through with it.

I: Yeah, probably because it was something I enjoyed doing **myself**; I just wanted everyone else to have the same feeling I had when I was doing it. (8:45)

R: I love your altruistic way of…

I: Yeah!

R: …thinking other people all the time

I: Yeah…[237]

R: Right! What other kind of interest did it have [the Sierpinski carpet problem] in mathematical terms, do you think?[238]

I: It was just some… I don't know why it was so interesting, it just… looks fun and I just wanted to see whether I could do it, whether I could get an answer[239]

R: Was it important that it had a formula there?

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[237] Ivan doesn't notice –or just ignores– the slightly satirical tone of my voice.

[238] In other words, which other mathematical aspects of the Sierpinski triangle he might wanted to promote with his presentation.

[239] With these statements Isaac confirms some of my remarks on his general attitude towards his task (see the file *Reflection on Isaac's presentation*).

I: **I felt that I had to have a formula or, I felt everyone had to get to a stage with a formula**,[240] that's why in the end of the thing I wrote up results for the people who didn't get it and work through how to get from results to a formula; I didn't give a formula, I gave them the results for what they would have got if they had… did it… like… got the right areas or minus the right areas off, and then they worked through the results to get a formula, rather than me just giving it to them. So…[241]

R: You told me, I remember, "if you choose the other way you'll end up with a nice formula"…

I: Yeah, yeah

R: …giving me a "carrot" to follow.[242] How do you think a teacher model is for you? (10:23)

I: From my previous experience [when] you are going to a lesson, you'd normally have something on the board which you would write down in a notebook, then you'd work through questions. And I actually don't mind that form of teaching because that's how I learned **the best**, from my experience. But this[243] is less about learning what to do; it's more about building a foundation **around** what you already know, to help you apply it to other things… rather than just to a simple question.[244] So…

R: Would you say that this is your perception of the whole course with this teacher?

I: Yeah, I think that's how I feel about most of the things; it's stuff I'm only using… **it's the information I've learned from learning maths but, I'm using the bits of information I've learned, I'm reinforcing it with other bits of maths I've learned, to make things that I've not looked at before.**[245]

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[240] My emphasis; a second important aspect in Isaac's choice of Sierpinski's carpet was that it involved formulas—see also footnote 2.

[241] Whenever Isaac says "so…" at the end of his talk, it means that this is all he has to say.

[242] That is tempting me, offering me a motivation in order to alter my approach towards the objective of the activity—as he saw it.

[243] Isaac refers to the teacher's sessions ('Nature of Maths').

[244] His comment refers to the teacher's course.

[245] My emphasis; Isaac describes how he keeps contact with earlier constructions while exploring new situations.

R: What is now your general picture of the NoM sessions that you had here?

I: They are my favorite lessons

R: I would like to understand better what you mean by that. Is it because you don't have to memorize things, because you have a good time or…

I: It's **because I enjoy doing it** rather than… **I'm enjoying maths** but **I really enjoy this type of maths**:[246] **thinking for myself rather than just playing [with] an equation to a question and getting an answer**.[247] That has its own merit to me because you get a solution which is a **definite** answer, generally at the end; **whereas this you can go anywhere, get right answers… no matter what you do![248] As long as you understand…There are certain things which you could do wrong to get wrong answers but, there's less… opportunity to…[249]**

R: because you are choosing the way, is that what you…

I: Yeah, you choose what you're looking at, so you choose what the answers are!

R: And is that exciting for you?

I: It is nice

R: It is nice…

I: It's nice, yeah

R: Do you think that it has enhanced or changed or altered in any way the conception of maths that you had… before… the construction of maths?

I: I don't know

R: Has it influenced it at all or… is it the same as it was before the sessions?

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[246] Ivan's emphasis
[247] My emphasis
[248] The creative aspect
[249] My emphasis

I: Oh, it's made me want to do **these** more! Like, before I started the NoM sessions I probably wouldn't have bothered spending my own time to work on something like this

R: Like Sierpinski's carpet

I: Yeah. So, after doing the NoM sessions I come across Sierpinski's carpet in a book and it made me… the knowledge of what we do in *Nature of Maths* made me want to work out what we'll do for this. And **how it could be looked at and how it could be furthered**…[250] like, instead of maybe doing this Sierpinski's carpet we could do Sierpinski's triangle, and rather than doing… I think it's called a hypercube, you could do a tetrahedron type one, where you got a tetrahedron then you remove the octahedron; from the remaining ones you remove an octahedron. So…

R: So does it give you ideas about new tools… in maths… to make it more interesting?

I: Yeah! Interesting... I find maths interesting, but I find this more interesting. But this is maths still so, technically …

R: Sorry?[251]

I: I find maths interesting, but I find 'Nature of Maths' more interesting than maths. But 'Nature of Maths' is still maths so, 'technically'.

R: Yeah, yeah

I: It doesn't make sense[252]

R: Could you for instance tell me which have been your favorite activities that we had with…

I: Nature of Maths

R: Yeah.

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[250] My emphasis.

[251] I asked him to repeat what he said because I did not quite understand it.

[252] He means that "Technically" (formally) his phrase is contradictory.

I: The ones I rove more around were algebraic… equation; getting to a solution so… I liked the… I think it was 'doubling the modulo'?

R: Oh the ring there, the…[253]

I: The ring… I found that quite interesting, and the 'A - B lattice'. The one where we're getting from A to B on a sixteen point grid; do you remember that one?

R: …

I: We had a grid like that [he is drawing it], we had A and B…

R: Oh, different ways of getting there.

I: Yeah, yeah and, that one was quite nice for me because I thought about a different way to how some other people thought about it, like a… like today with… I think it was… Sinead? No,… Elena, Elena [pseudonym], did the… "get the fly from A to G".[254]

R: Yeah, yeah

I: I did the same that I did for the 'A - B lattice', modeled out A and gave each of these letters; and I went to see… F from F I could go to G or H… and that got me to an answer, and it was **a definite answer**; and, I found that kind of nice.

R: Mmm…

I: I didn't go to how it would change if B was in a different place, I only solved it for where B was; but you could take it a lot further, you could change the amount of points, [also] where it is on the thing. I'm not quite sure whether you'd get an algebraic answer.

R: Doesn't it sound a bit like the Decision course that you got?[255]

I: Yeah, I'm not too keen on Decision; **I'm not too keen on Decision but I like this![256]**

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[253] Ivan's and other students' findings in the *Doubling modulo* task looked like rings.

[254] Ivan refers to Elena's presentation that took place the day of the interview.

[255] I am teasing him a little bit, since he had already told me that he does not like the Decision course.

[256] He responds to my teasing.

R: I haven't found anybody who is! [keen on Decision][257] Okay, let's put it in a different way, which one was algorithmic… algorithmic procedure, first you do this then you do that then you trace all the different ways or possibilities to get to the result you want?…

I: Quite a lot it was… algorithmic

R: [it was] Luck [pseudonym], how to get through the bridge

I: Yeah, that's a…

R: First you do that then you do this and then you do… there is a pattern

I: Counter-intuitive one, very counter-intuitive

R: What did you think of this activity?

I: I've seen it before! I try to remember when and I remember it was a counter-intuitive thing-process as to why it would work; which was probably why I think I was the first person in the class **to get a definite answer.**[258] So, if that kind of helped;[259] as well as it was Rosie's that I knew a bit of stuff about what she was doing.

R: Did you?

I: You know, when it was φ, the golden ratio,[260] working back from a number; do you remember that?

R: I don't know what you are talking about

I: Robyn's lesson?

R: Oh, Robyn's lesson, Fibonacci sequence yes, yes

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[257] I had already an interview with Tom [pseudonym], who was feeling the same as Ivan about this course.

[258] My emphasis this time; it seems like it is an important element for Ivan, since he is choosing the same expression.

[259] The phrase was said in a humorous way, possibly wondering if his answer to my question is 'sufficient' (for my research).

[260] In mathematics and the arts, two quantities are in the **golden ratio** if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one $\left(\frac{a+b}{a} = \frac{a}{b} = \varphi\right)$. The golden ratio is an irrational mathematical concept, equal to $\frac{1+\sqrt{5}}{2}$, and it is also the limit of the ratios of successive terms of the Fibonacci sequence, which was the principal idea in Rosie's [pseudonym] activity.

I: I had previously read about Fibonacci sequences and had produced the result for φ (4:06)

R: Which [students'] presentations-activities [do you think that] did stand out?

I: There was one that I really liked; would I just be able to look through my notes?

R: Yeah.

I: It was Carry's I think

R: Oh the boxes!

I: Yes, Caroline's boxes… there's a lot of work that you can put into Caroline's boxes and it seemed quite interesting. (7:15)

R: In which sense? Can you give me a short description of it?

I: It was the idea of how many arrangements you can have for a… I think it was 12 boxes?

R: One [up] to 10?

I: One to 13? [he is checking his notes] No, one to 11!

R: To eleven

I: Boxes one to eleven, and how you can arrange it, so you get two [columnes] of the same height; then the idea of how you can arrange it to get three of the same height, what sort of ones could you use for three for the same height, and…

R: So that's a rich activity to follow through, right?

I: Yeah! And, if we had more time we could probably get a bit more working out, probably get to *definite answers;*[261] if we had some time, if it wasn't very quick.[262] This is pretty interesting but… you can solve that after a few minutes.[263] I liked how Rosie chose a Fibonacci thing because I do like Fibonacci sequences, because I think

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[261] It is one more time that he is using the same expression.

[262] There was a limit of 20 minutes presentation plus 10 minutes discussion for every presentation.

[263] I think it was Joanna's [pseudonym] activity that he was referring to, while he was browsing his notes, trying to trace the presentations that he thought as the most interesting. *The challenging aspect seems to be important for Ivan;* see also what he says in contrast with Caroline's activity.

they are quite clever[264] the way that Fibonacci sequences are solved the golden ratio and how enticing… which practically I knew because I got to spend a bit of time messing around.

R: You are using the φ number to manage it [in order to find the previous terms of Fibonacci sequences]

I: Approximate φ number anyway[265]

R: Yeah, but that's how you dealt with it

I: Yeah, yeah

R: That's why one would get quickly to an answer[266] and then…

I: Yeah!

R: wouldn't be any point for him… for thinking. What about Mary's [activity]

I: I found Mary's quite interesting because… well, because… the idea…

R How did you get to the answer?[267]

I: Yeah, **how it works to an answer[268]** and then, thinking about **what you've actually done to get that answer,[269]** and **then how you could change it to do the same thing but have the same answers**[270] like… I changed what she

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[264] My emphasis; the challenging aspect is important.

[265] Approximately, since φ is an irrational number ($\varphi \cong 1{,}618033988749$)

[266] I rather instinctively repeated –with a low voice– what he said earlier (see footnote 172); he quickly responds the same way as before.

[267] He was the first to find a solution.

[268] My emphasis; it is again Ivan's expression of a "definite answer".

[269] Reflecting upon and analysing the method he just used in order to get an answer seems to be the next step after finding a solution, and Ivan sounds like he has objectified it, and it has probably become a habit.

[270] My emphasis; the reification (treating the method as an **object**) seems to follow the analysing process in his attempts to using the same expression.

[270] There was a limit of 20 minutes presentation plus 10 minutes discussion for every presentation.

[270] I think it was Joanna's [pseudonym] activity that he was referring to, while he was browsing his notes, trying to trace the presentations that he thought as the most interesting. *The challenging aspect seems to be important for Ivan;* see also what he says in contrast with Caroline's activity.

[270] My emphasis; the challenging aspect is important.

[270] Approximately, since φ is an irrational number ($\varphi \cong 1{,}618033988749$)

[270] I rather instinctively repeated –with a low voice– what he said earlier (see footnote 172); he quickly responds the same way as before.

[270] He was the first to find a solution.

did so, instead of timing it by 2 I decided to time it by 4, then add 7, then… I think it was timed by 25, then add [the] age, something like that and, I got **the answer** of that so I'd end up with a date, so… 0514, it would have been 0515 eh… 75 plus 18 is 93 so, that's what I would have got at the end and… from that I can work out a birthday and an age[271]

R: I don't get it really; oh it's the 14th of May yeah. Then 15th, how does it get 15th? Plus 1, you say…

I: Yeah, because I've got the equation 100d+1+75e+8

R: So you got the equation, then you tested it, is that what you're mean?

I: **I got the equation and from that equation I would be able to put my birthday at the beginning, and get that answer out**.[272]

R: But how did you get to the equation, that's the [point]

I: I just…

R: It was amazing, I thought

I: For Mary's one?

R: Yeah.

I: Because… I just followed through the steps. I wrote… I said date is d, and then she said she times it by 2, so it's just 2d, then she times it by 2 at 5; from that she times it by 50, so you get 100d plus 250. Which then you can rewrite if you think about the columns of what you've got, as in units, tens, hundreds, thousands, ten thousands. You've got a number, which is… oh what I need,[273] a hundred thousand

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[270] My emphasis; it is again Ivan's expression of a "definite answer".

[270] Reflecting upon and analysing the method he just used in order to get an answer seems to be the next step after finding a solution, and Ivan sounds like he has objectified it, and it has probably become a habit.

[270] My emphasis; the reification (treating **extend** the task; this paragraph seems to be a hint of Ivan's creation of **mathematical objects** and his approach to **mathematics** (in particular to problem solving). Cf. the footnote 194.

[271] He is explaining how he found the solution to Mary's activity, using his birthday and age.

[272] My emphasis; Ivan is exemplifying my interpretation of what he just said in footnote 192; putting his "birthday at the beginning, and get that answer out" is meant in terms of testing a more general impact of his equation. We notice that what we called reification process in footnote 192 is 'instant'.

[273] Ivan is actually solving the problem on the spot.

as well… Then you've got a number which will cover these 4 columns, which would relate to your date… so 100d, so… the bottom number would be 4, 1, 0, 5,[274] that would be 100 times my birthday; but 100 times my birthday plus 2 so it would be an extra 200[275] so I'd get 0, 5, 1, 6,[276] which is what I got out of there. And 50 plus my age,[277] because my age is a two digit number, I'll get the 18 plus 50 so I'll get the 68 there. So from using the knowledge, that's all I have to do, is… I just started off with another number, eh… with another date, decided what else…

R: That was brilliant actually, yeah… You followed a very unique way of doing it I mean, me and Lewis were doing it with many variables there…

I: Yeah

R: …and it would take more than twenty minutes to get to the answer…

I: Yeah

R: …and then you came with the [solution]…

I: Unfortunately people don't understand my explanation or how I got to the answer; which is always a bit of a nuisance. So…

R: You *think* you've said enough for people to understand but people need more; it's not enough for them

I: Yeah! Because what I write down is a thought process that happened at my head, just me trying to write that thought process down; sometimes that doesn't come out too nicely, and people can't follow it.

R: All right! I don't think I have another question to ask you. Yeah, that's it

I: Is that it?

R: Yeah, now you can return to your 'favorite' session[278]

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[274] He means 1, 4, 0, 5 (the first 2 digits for the date, the other 2 for the month)

[275] He means that the previous number needs the addition of 200 in order to produce the correct date (see the next footnote)

[276] He means 1, 6, 0, 5.

[277] The number 50 (out of 250) is added to other two (last) digits, which Ivan has mentioned; these digits are related to the age.

[278] The *Decision* course.

I: Ohhh, what!

Appendix C. Diana's lengthy interview, 15/4/2011

Diana seemed to be the less active member of her group during the NoM sessions, but she usually achieved to manage the activities in her own way, approaching them with quite different methods than all the other students. This interview was partly motivated by my attempt to investigate the mental constructions that led her to these practices.

I had asked her to bring the notes and homework from all the sessions, and she brought all of them.

Diana: This is everything I've got!

Researcher: Alright! I brought some stuff too; in case you would not bring something I want [to discuss with you]

D: Well, these ones here are the ones I was thinking of doing to hand in for my course work, and then… all of **this** is just what we've done in class [she seemed to be proud of the volume of her 'production']

R: Alright!! [it was indeed an impressive volume of work, especially the class notes]

D: So, the ones I've got here, the snooker one, that I've… [she was browsing her papers on the snooker activity] kind of started to write up, but then not really gone much further… with it

R: So at first you produced it in the computer, I see, and then…

D: Well no I… I did all my… I did all my notes here, so these are all the… I think this is all. Because I gave the teacher some of my notes but he can't find them;

R: Oh really?

D: Yeah [laugh]

R: Did you give them to him, literally, or…

D: Yeah

R: …was it photocopies

D: Oh they weren't photocopies; I should have given to him photocopies really… I think… yeah that's it; so, what I did **here**

R: [you mean] in the snooker [activity]

D: was… we started with the 4 by 3 [case]. I think…

R: [you mean] 4 columns and 3 rows

D: Yes! And then, I extended it by looking at 4 by 2, 4 by 1, 4 by 4 and all that sort of thing

R: Keeping 4 as a standard there…

D: Yeah, keeping 4 as a standard, and I went up to… I noticed there was a connection with anything that was a multiple of 4

R: Really?

D: So, when I… I sort of stopped at 4 by 8, but then to test my theory I checked 4 by 12, 4 by 16, 4 by 20 and 4 by 24. And I think I can… it's been a while since I've looked at this one; but the connection… because what I did when I wrote it up [was that] I looked at the number of lines, the number of times that it hit the side [i.e. the bounces]

R: Yes…

D: I looked at where it started from and where it finished (2:30) and then I wrote observations about what I noticed.

R: The starting point is [always] the same, alright?[279]

D: Yeah…

R: You didn't look the starting point… a different one,[280] the same one all the time, yeah?

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[279] I was surprised that she was writing down the starting point of the ball in each case, since the activity (as it was set by the teacher) indicated that the starting point was always the bottom left corner of the snooker table.

[280] I try to understand what she did during the activity, since she is showing me a table in her homework (the 'final' outcome), where she keeps record of the starting point, although it is supposed to be the same all the time.

D: No, I could have done but instead I chose to alter the size of the grid. And I suppose it's something that I could start changing the starting point I never thought of that[281]

R: I think that it would have been a different problem,[282] wouldn't it? I don't know, what do you think, was it a complicated one, was it a manageable one, was it a…

D: No! It was manageable I was doing really well with it! But then… because I did 4 by something then 2, I kept 2 as a constant, then I kept 3 as a constant, 5 as a constant, 6 as a constant; and I found a rule[283] for… if you named the size of the rectangle x, y

R: If you… what?

D: The size of the rectangle, if it was a 4 by 3, the 4 would be the value x and the 3 would be the value y; for example this would be x and this would be y.

R: [That is] The columns would be the width and the rows would be the…

D: The height?

R: Yeah[284]

D: Basically the x would be constant and the y would be changing

R: So the y would be the variable

D: Yes!

R: And the x would be the one that you would keep as a constant

D: The reason that I did that was because I **noticed** that if the y value was **odd** then they all finished in… you know I could predict where they were going to… which side[285] of the table that it was going to finish at. But I didn't really… I haven't looked at this one for a while actually. So I was doing quite well with that but then

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[281] At the table that she had written the starting point was mentioned in every case—although it is supposed to be the same all the time—but she was just recording it rather than questioning it.

[282] In the sense that the teacher had specifically asked to use a particular starting point, although the same generalisations would have been produced.

[283] It is interesting that Diana is using the expression "I found a **rule**" to describe the generalisation process.

[284] The number of columns is the x and the number of rows is the y.

[285] She means the **hole** of the snooker table.

I stopped for some reason because I started doing something else [laugh] So I need to look further into this before I can obviously handed it in

R: Did it give you pleasure this activity, this particular one? Did you like it particularly? Because it's the first one you mentioned [and] that's the only reason I am asking you

D: Well, the only reason that it's the first one I mentioned is that it's the first one in my file! It is something that interested me initially because it **is** something that you can **easily**[286] find patterns for; so when you were looking 4 by something, for example, it was easy to see patterns in it. So… you can see by just doing the 4 by 1 up to the 4 by 8: most of them with the exception of 2 all finished on the right hand side of the snooker table. So if they started in the bottom left, then they world finish on the right hand side either at the top or the bottom. So, I **liked** that because it was something that I easily found patterns for. But it's just making it a bit more… **general**[287] I suppose for different sizes of rectangles and… and things like that, so…

R: Do you feel that there are more powerful… eh… patterns[288] that you could find if you would…

D: If I was to put my work into it **yes** and I'm planning on doing more work on this one

R: Why? You like it or…

D: Well it's… one[289] because I've originally done quite a lot of work on it…

R: Already, yeah

D: And I wouldn't want to disregard this one in my course work; because I've already done so much on it. And it **is** quite interesting just to see the different

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[286] All the words in bold indicate the parts of her speech that **she** emphasized.

[287] My emphasis; it is the first time that she is using a word relative to **generalisation**, after the use of words such as "rule" or "patterns".

[288] I avoided using the word **generalisation,** in order to minimize my influence of the vocabulary that she was using.

[289] She means 'firstly'.

patterns[290] and how you get them, and the different… the different patterns that you get with the **lines** when you… when you are drawing them

R: And you do find patterns all the time? Or do you have hard time finding any patterns?

D: Not necessarily, no, I found this one… I found this one ok; I think this one is one of the more enjoyable ones that I found of what we've done. I don't know **why** but… I can't find a **reason** on it but…

R: Yeah… yeah, right[291]

D: It's that it's something I've been able to…

R: The snooker

D: To understand and…

R: And manage and…

D: Yeah… I've got [laugh] **lots** [she emphasised the word 'lots' while browsing through her papers]

R: Right! You've done a lot of work, I noticed!

D: I know!

R: Even during the sessions you were working hard!

D: Yeah, yeah!

R: I have noticed that!

D: Yeah!

R: What do you think about them generally, do you think that they are easy, tough? (7:30)

D: I don't know, sometimes I feel as though people… I think everybody else seems to understand it more than me; like with **the doubling one,** everybody understood

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[290] She keeps using the word "patterns" in order to describe her findings, possibly because I used it (see previous footnote).

[291] I agree with her in the sense that she doesn't have to give argumentation for her preference; that our conversation may move on to a different topic.

it straight away and they could just do it as a loop, like this and they understood what the next number would be but I could not process that. The only way I could do it was by doing a flow… I invented up doing a flowchart for this, like a flow diagram[292]

R: Like what?

D: [she searched her notes and she showed me her work during the task—see footnote 14]

R: Oh, that's interesting![293] It's like an algorithm!

D: Yeah!

R: It's like a programme in a computer: "Do that", "YES", "NO", "YES", "NO"[294]

D: Yeah! **Because this is the way that I understood it!** Because everybody else seemed to be able to look at the numbers and know what the next number was!

R: What made you think of this way of doing it?

D: **Because to me that is quite systematic![295]** And I… **I respond to things that are systematic** and that I can… **I know** what I…

R: What do you mean "this is systematic"?[296] What is 'systematic'?

D: Well, with this…

R: This you mean… what?[297] Your [notes]…

D: This is all my working for it

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[292] She is talking about the flowchart that she did later at home, not about the algorithm that she devised during the task.

[293] See Diana's coursework, where the final form of her construction is depicted; the initial one had the form of a computer programming algorithm, very close to a computer programme written in a pseudo-language (a sequence of orders that would inevitably produce the necessary results).

[294] The 'doubling modulo' activity was one of the topics that I wanted to discuss with her, but she mentioned it soon after the interview started.

[295] 'Systematic' was a term that the teacher had often used during the sessions and it seems like that she had picked it up.

[296] And I was pointing at notes that she kept during the task.

[297] Sometimes, like in this case, I make questions in order to be clear for the recording what we are talking about.

R: I was there, I remember it, yeah!

D: So I did, when I looked at the number 9… I knew that I had to… once we'd found the rule[298] for it, then… like I say everybody else seemed to be able to say the next number but I couldn't, so I was having to device this kind of system, where I've said "right, okay, if I do **this** to that, if I times this number by 2, is it bigger than the 9? No it isn't! So what do I have to do to it? Then I have to do this to it". And, that was easy for me, it was very very long-winded and it took me **a lot** longer to understand it than it did for everyone else. But now **I've done this** and **it's my way,** nobody else seems **to have done that way**. But this is, this is how I have understood it (9:30)

R: Right! And from this[299] you were drawn to the… the flowchart?

D: Well, when I was looking at this, I was thinking if I was to give this to the teacher, he'd look at it and he'd be thinking "what on earth has she done?". Because he would not understand it! So I thought the only way I can do it is by doing some kind of flowchart, flow diagram or algorithm or whatever to be able **to explain my steps in the thought process that I went through, to be able to understand this activity…**[300]

R: The flowchart is much more advanced than this one, although it might describe the same thing, but the flowchart is far more advanced, I think

D: Well, **I knew** that if anyone else other than me looked at this they wouldn't know what was going on

R: I did

D: Oh, did you? Well I just **assumed** then, shall I say, that if **anybody looked at this** they wouldn't [know] what was happening and how I'd got my answers; which is why I did that [the flowchart][301]

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[298] Another use of the word 'rule' (see footnote 5).

[299] From the 'sequence of commands', the algorithm.

[300] My emphasis.

[301] Diana is emphatically repeating that her **motivation** for advancing her initial 'sequence of commands' to a flowchart was **the need to be understood by the other people.**

R: That is far more sophisticated

D: It's a little bit confusing though, isn't it?

R: Why?

D: It's easy to follow but there's just lines everywhere [laugh]

R: [laugh] is it your thought depicted here?

D: Yes!

R: Is this precisely how you have to think and how you thought…

D: **That's how I thought** to be able to do it

R: It's a bit of a maze, isn't it?

D: It is [laugh] but...

R: [But] There is an outcome, somewhere, isn't there?

D: **To me that makes perfect sense!**

R: Yes, yes, absolutely, I like this one… I'll photocopy it actually[302]

D: Yeah!

R: I like it very much

D: Well, I've got on my… I can e-mail it to you, if you like

R: Oh please, yes

D: Because I've got it on my computer; so if it's easier I can email it to you

A: Oh yes, that would be even better

D: I still need to double check it because again it's a while ago since I've looked at this; and I did double, triple and quadruple checked that this worked and it did, but I'll double check it again [laugh], just for my own benefit

R: Alright, I would accept it as it is, even if it had errors,

D: Yeah!

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[302] My statement was formed as a question actually.

A: I mean it doesn't make a big difference to me; to me it is the construction that matters

D: Right, okay

R: And I can understand what you're trying to do here; so, that was about it with this activity, which I call it 'rings'… the doubling

D: I think it was called 'doubling', yeah…

R: Doubling, yes, 'doubling modulo'.

D: Yeah but the reason… **I did quite enjoy that one but that was because I had a different way to everybody else**; **that's why I liked it** [laugh][303] (12:20)

R: Do you usually have a different way to everybody else or do you usually have the same way with somebody else?

D: No, usually my thought is… my thought is the same as everybody else's, so **this one quite intrigued me** because I thought it in a different way than nobody else seemed to, so it intrigued me, because of that.

R: It did stand out actually, I remember [that] I came and said "what are you doing here?" and you said "Oh, that's **my way** of understanding it"; you kept working on it

D: I was way behind everybody else but… well that's all of that

R: Anything else about this activity that you would like to mention?

D: I don't think so, not particularly. I think this one is finished actually, **that's my end product**;[304] so I think that one's done! But I've just explained how I understood it[305]

R: How did you feel about the other students' ways of understanding it?

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[303] The short laugh comes after a statement that was uttered very seriously.

[304] She talks about the file *"Danielle_Number Sequences" document*, which she actually sent me.

[305] See talks about the final three paragraphs of the *"Danielle_Number Sequences" document*.

D: I still don't understand how they do it; because a couple of people tried to explain it to me but I was just completely lost, I didn't know how they could look at a number and go "alright that' the 1 so the next one needs to be 2 [or] okay that, the next one needs to be 4". I still don't understand it, their way. Everyone was doing the wrong thing[306] I don't think it was explained **properly** to me, which is why I personally haven't understood it

R: Yeah, yeah

D: I like my way! [laugh] **It's a long way, but I like my way** [laugh]

R: [laugh] Okay! You don't need to understand anybody else's way, is that what you mean?

D: No, I don't need to

R: I thought that you are kind of finding you own ways anyway and sometimes they get along with other people's ways, not that you are doing what the others do.

D: This is the 'Garden-Path' one. (16:22)

R: Right

D: And there's quite a lot of work on that one as well, actually. This is explaining what we did in class

R: In class

D: This actually will be in colour when I'll print it out properly; and then, I need to look into this **more** as well because… where is it? [she is browsing her notes] I started looking once we… because we looked at gardens that were rectangular in shape, and I started looking as to whether there was a difference in the area of the… in the path around the garden if it was landscape or if it was portrait. So I started looking to the difference in that; and then I found something for that, and then I started looking at if the garden was an 'L' shape, and I was going from there… and then… because **what I did:** when I was looking at the 'L' shape I was looking like a square path to go around it, and I did loads of work on that and then I realised

[306] She means that the others did not explain their way of thinking in a way that she could understand.

that actually, I don't need these bits here because I can just do a path directly around it, I don't need the square; and I just… I still put the work in because it seemed a shame to have done all that work and not to include it. So I've just included it all [laugh]

R: [laugh] So you include everything, anyway

D: I just included it all, why not? [laugh]

R: Maybe you are not convinced from your new discovery, your… discovery; maybe you still have to check it out (18:00)

D: Yeah, I think I've got a bit more work to do on it, but I managed to… because I looked at **predicted formulas** for it, so I did a few predictions and then I've looked at… I've gone further into it to see if my predictions are any good

R: What did you find?

D: I don't think any of my predictions were any good[307]

R: Oh really?

D: I've looked at them, they don't work and then… I thought a formula and how I can… I've seen that the formula does not work if you will; and then looked to see if I can change it in any way. I've still got more to do on this

R: Which are your favorite activities?

D: I liked the 'New York cop' one and I liked the 'tessellation' one as well

R: The tessellation one

D: Yeah, the one we did with the pentominos and the…

R: Alright

D: I liked the 'New York cop' one because again that was something that I seemed to find… **not necessarily a formula for but I found a pattern**; **and once I**

[307] It seems that the 'trial-and-improvement' method is one of Diana's favourites.

found a pattern it was really easy to move on to the next stage[308] and… but I haven't looked at this one for a while

R: And how did you find the pattern?

D: How did I find it? Well… [she is browsing her notes] So this is the size of the precincts if you will, what we were looking at –I've got lots of drawings to go with this- because at first, **the first thing I noticed before I got any kind of formula for the number of police that were needed, the first thing I found was actually a pattern in the drawings**.[309] When I was doing the drawings, I found **patterns** in them; **and then I started playing around with the patterns**[310] and whether it would be any less[311] if you moved the position of the first cop and… **and I started finding a lot of different patterns with it**.[312] But I've got **so much** work on this... But then I got quite frustrated with this one because I found a pattern and then I tried it with something else… what did I do? I found a pattern and then I tried it again but then I realised that I… I can't remember what I did! I had to start writing things as I was doing it… **Oh that was it!** Originally when I was doing it I was looking at, like this pattern here! If you can see it's kind of like **a cross**; if you look the pattern that it's made, there…

R: Oh, yeah, yeah

D: And then there's one [cop] in the middle [of the cross],[313] so I started looking at that and going further into that; but then I realised that -I can't remember how I realised- but then I realised I can actually do it like this… See? **It's all in diagonals**

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[308] My emphasis

[309] My emphasis

[310] My emphasis

[311] Necessary cops; she is describing the way that she was trying to find the least number of policemen that were needed in order to survey a number of precincts , as the task demanded.

[312] My emphasis

[313] Diana refers to the teacher's drawing on the board at the beginning of the task (see fig. 1)

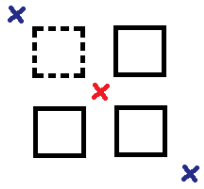

Figure 1*:* The teacher's drawing at the beginning of the session; the dotted square and the cop in the middle were drawn first, while the other 2 cops were suggested by Donald, after the teacher drew the other three squares.

R: Alright!

D: And that [pattern] gives a lot less than this [the cross pattern]. It gives a lot less [cops] than this. Oh, maybe not **a lot** less, it's 17 there and 16 there

R: For the same shape [for the precincts] it gives you less cops?

D: Yeah, **for a different pattern**. So, **rather than looking at the numbers I was more concentrated on the pattern that I could see and then once I was happy that I had the optimal solution for it, then I started looking at the numbers to see if I could find… any kind of… pattern… any patterns in the numbers and then started trying to work on a… a general formula for it but I've struggled with the general formula because of the differences;**[314] because I've got a pattern and the difference of the numbers of cops is 1, 2, 1, 1, 2, 1 ,1, 2, 1 ,1, 2, 1 ,1,[315] and it goes on and on like that… but I need to… work on it to try and find a general formula for it. So that's something I'm still working on

R: So you also think that it's a 'rich' activity, that you can take it further, I suppose

D: Yeah, yeah

R: Since you say you have to look at it further

D: Yes! Yeah, I need to look into that one further (23:01)

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[314] My emphasis

[315] Diana and her group arrived at the diagonal pattern and the same *number sequence* that Ivan found; but she and her peers did not manage to advance these findings to formulas, as Ivan did, since they were still uncertain of the diagonal pattern that they ought to choose and they could not arrive at a unifying pattern (see Diana's coursework).

R: What about the activities that **the students** presented?

D: I liked… I liked the one that Caroline did, which was the one with the box, I liked that one and that might actually be something I might look further into, to hand in with my final sort of portfolio work; because I enjoyed that one.

R: How did you treat it? What happened there? What did you do there with it, during the activity?

D: Well, when we first started looking at it… when we started doing the activity… like I say everybody had different ways because we were splitting in two teams, so the team that I was in we were actually playing with the boxes, we were not looking at the numbers, we were playing with the boxes to see if we could do it. But then, the other team… I suppose… thought… I don't know how they thought about it but they… because you had to make two towers, didn't you? So I did all the numbers together divided by two and then realised that you can't make two towers because it was an odd number, so it was impossible;[316] so **they did that** but my team weren't thinking about the numbers and playing with the blocks. So **it was quite interesting to see how different people view different things and the way they were doing it**;[317] so then after that we were given an extra block so we had eleven blocks…[318] and then… when we were looking at that… eh… what did we do when we were looking at that? Then it was the case of looking at all the different variations of how you could do it… so, with my team I started looking at the… because we knew that … when we added all the numbers together we knew that each tower had to have a value of… I think it was 33 or something like that, so we started looking at the different ways you can make 33 from the 11 numbers but obviously there are quite a lot of combinations for that; when I started looking into that one of the other girls instead of starting at 11 she went back and started looking I think at… at 6 if there were 6 blocks? No… I can't remember, she started looking

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[316] She means that it was impossible to divide the odd number (the sum of the boxes' numbers) in two equal numbers (blocks).

[317] My emphasis.

[318] She means boxes (11 boxes with the numbers from 1 to 11, instead of 10 boxes with the numbers from 1 to 10)

at smaller, so rather than being 11 blocks she may be looked at what would happen if it was 3 blocks to see if she could find a pattern that would then maybe lead her to the answer for if there were 11 blocks.

R: Alright!

D: So I started from the 11 and was working on that, and she started from if it was smaller to see if she could find a pattern

R: And why was it so interesting? Did you come to results or did you…

D: Well no I didn't because we run out of time, which is why I was wanting to look further into it just to see; because I again I had a system of… I was looking at the… I was looking at the blocks and I was… because we had the numbers [up to] 11 I was saying "right, okay, so if we had 10 and 11 together that's going to give us 22, so how many of the **remaining** numbers will give us the combination of… of what we need, of the 11 we need left over"; so I was looking at **that way** and… as I say Barbara was, I think it was Barbara that was looking at the smaller blocks

R: Which other one interest you? Which other activity was the most interesting for you?

D: Out of everything we've done?

R: The students

D: I quite liked Ciera's [activity] where you had to find the triangles, where you had a shape; she gave us a pentagon and you had to find how many different triangles were in it. Again that was something that I didn't feel as I finished… because **I feel as I work quite slowly and… everyone seems to get it like that**.[319] And I was thinking "but I don't understand how you've got this and where have you got that, from where have you got this from". So… again that has interested me because… you know, you're just looking for the shapes and looking for all the different and you are seeing if there's any patterns and…

R: There were many activities that were interested in the shapes like Dave's

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[319] My emphasis; what she means is that everyone seems to get it 'instantly', quite faster than her.

D: Yeah, I liked Dave's as well

R: What is it exactly that draws you to this kind of activities? What is it, the colours, the drawing, is it something beyond that… also? (28:00)

D: I'm not really sure; it's just something I've always been drawn to, even as a kid; it was always them kind of activities that interested me and I don't know what it is about them, I don't know whether it's because… I don't know, maybe because **of the person I am I like to be able to… because whenever I do anything like that I can see how it's systematic**,[320] how, okay, with Dave's [activity], I've drawn this square so what are the variations of that that I can do; and then I find another pattern and I think "alright, what are the variations of that that I can do"; and **to me, it's something that is very systematic and it's just personally something that I like and being organised and… to me it's organisation and it's what I like**

R: What about Tim's? Did you like Tim's [activity]?

D: No

R: Why?

D: Because… At first I didn't understand it; it put me off at first because I don't know anything about chess I've never played chess I just don't know anything about it. And it's not something, I don't know why, but it's not something that's particularly interested me, before. Because it's a lot about forward thinking, isn't it? And I can't really think…

R: Forward thinking?

D: Well, from what I understand of chess a lot of people have said to me that you think two moves ahead of what you're actually doing

R: Yeah, but that's because there are many different pieces; there [in Tim's activity] you had **one knight**

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[320] My emphasis.

D: Oh no, I understand that but this is what… that's why it put me off but then I did it! And it was okay when I did the first one, when we had to find where it would move in the first instance. But then I think it was a little bit confusing, the instructions that Tim gave to me were a little bit confusing from there; because I was saying "okay, so from this place, here, this could be my second position but it has no account because it overlaps with this". And I did not fully understand

R: Oh, the knight can go over [other pieces], it doesn't matter if there is something on the way

D: And I did not fully understand it

R: There was an understanding that you needed there. What about Isaac's, how did you feel about Isaac's, Sierpinski's carpet activity

D: It took me a while well, **it takes me a while to do everything**… it took me a while to understand that but once I did… because Isaac was very patient with me [laugh] but once I did understand it I really liked it.

R: How did you treat it? I don't need details I mean generally

D: At first I was looking at it and I was thinking that you'd have to draw it to be able to understand; because **for me, usually I have to be able to visualize things to be able to understand it**[321]

R: You mean to **actually** visualize yeah?

D: Yes! So I was drawing all of these [the squares within the square]. Well I didn't draw that one.[322] I drew the 1st, 2nd and 3rd stage; which to me was how I understood it and… as I say once it was explained to me, and it was just for me getting my head around the fact that… this bit here… because we had to find the area of this didn't we? So then… once… and I understood that… obviously then we look at the separate ones, so you had to look at the whole thing of the… the area of this, with these bits missing; and I understood that but when we got into this I started to get lost, when we got into the 3rd stage I started to get lost! **But then**

[321] My emphasis.

[322] She means the 4th stage; see *'Reflection on Isaac's activity'* file.

Isaac came over and explained it to me[323] and he said "well this is this and this is like that and like that" and "Alright, yeah…"; so **I really liked it once…**[324] [I did understand it]

R: And what did you do after you understood what's going on?

D: Well because I'm so slow in understanding things we've moved into the next person's…

R: No, no, no not what Isaac did, what did you do? I think Isaac told me in the interview that I had with him that you were treating one part of the shape, trying to find the area for this particular part and then trying to come from this part to the whole…[325] Is this so? I mean that is how he described it to me

D: Maybe, maybe that was what I was doing; maybe that is why I confused myself

R: Because I thought "Oh, quite typical for Diana"… [326]

D: [laugh]

R: She is getting fixed into a point

D: Yeah…

R: that she can manage and then she…that was my [impression]…

D: Yeah…

R: is that what…

D: Well, originally, for some reason… because when this was drawn on the board we had the grid in it, didn't we?

R: Yes

D: So this was a separate one and this was a separate square, so I found the area of that but then it didn't click with me straight away that we were still trying to find the area of this; I thought then we were looking at the **smallest square** and we

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[323] My emphasis (see also Ivan's description of Diana's reactions in the '*Reflection on Isaac's activity'* file).

[324] My emphasis.

[325] See *Isaac's transcribed interview 070411* (8:00 min)

[326] See *Isaac's transcribed interview 070411* (8:00 min)

were trying to find the area of that! And then with this one looking at that sort of smaller square that would be made there and trying to find the area of that smaller bit. So **I think that's what I was trying to do at first, look at the little bit and then expand it from there**[327] but obviously that's not the way to do it [laugh]

R: [laugh] I don't know if it is, I mean why not? You can find a way through it

D: Yeah…

R: That would be okay, wouldn't it?

D: Yeah…

R: You found [a way] with the other stuff there [the doubling modulo activity]. Right! What about Lewis's [activity]? What did you think about that?

D: I don't like things like that!

R: The bridge and the…

D: I don't like things like that because I never understand; my brain just won't allow me to work in that way… Because I never… Because… I don't know why but I've got a… there's a game that I've got that has lots of different puzzles like that and I… I don't know why but **I've never understood them even as a child**; **they've completely baffled me** I've never… and I don't know what it is about them **I can't explain what it is but I think it again the only way that I can explain it is going back to my system, so I would have to say "right, okay, well that one is not going to work so what could I do next? And then what could I do next?"**[328] **And I think that the problem that I find is because I have to work in that way… people work faster than me and I hear what they are saying and I hear them shouting answers so I think "that must be then!". So I'm not working out for myself I'm just overhearing what**

[327] My emphasis; it is also a strategy that the teacher had suggested during the sessions.
[328] 'Trial-and-improvement' method.

the people are saying because other people work faster than I do[329] (35:00)

R: I didn't get that actually…[330]

D: With Lewis's [activity], it was the one with the different times of people. It was 5 people walking…

R: I've got it here

D: Yeah… And they only had 30 minutes for the lantern, didn't they? And you could only walk as fast as the slowest person, couldn't you?

R: Yes

D: Well, when I started to do it just because of the way I work and I always have worked I would have to start with one number so I started with 1 and I said "okay, if he went over then, who would have to go over next and who would have to do this and who would have to do that"; and it's a long way to do it but it's just the only way I understand it. But with the others because they worked faster than me I heard someone say "the number 12 has to go across"; no, not the number 12, "the number so and so has to go across first", so I think "the number so and so has to go across". So I'm not thinking for myself at that point I'm just hearing what other people are saying, I'm thinking "oh that must be right! So I'll start with that and then go from there". So then I'll start with whatever number they have said and started **my system**[331] again; but then I'll hear someone else say "oh then you have to take that one over". **But because I cannot work fast enough with things like this I never get to the answer myself because I always hear what other people saying**[332]

R: So you are changing your mind they change your method while you are processing it, is that what you mean?

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[329] My emphasis.

[330] She was speaking rather fast and I wasn't sure that I understood what she was saying.

[331] My emphasis.

[332] My emphasis.

D: Yeah… sometimes with Nature of Maths I never felt as though I could keep up and work as fast as everybody else because a lot of people seem to get things [just] like that[333] and I've always felt as though I was _[334] behind (37:16)

R: They have their own problems! I think that you think of them as something sublime; that they are doing everything so fast

D: They are though!

R: Are they?

D: Yeah!

R: They have problems also!

D: I know but they're all… I don't know they just all seem a lot better than me, at maths

R: Oh that's because all those who have problems don't speak…

D: [laugh]

R: [laugh]

D: I feel as though I'm just sat there all time but I haven't got a clue; I don't know what's happening, I don't know what's going on and then when I always feel as though I get to a point of understanding and we've moved on something else!

R: That's happened to me also!

D: Yeah…

R: It's natural I think

D: Yeah…

R: What about **your** activity? (38:00)

D: Ohhhh… [a grimace of discontent]

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[333] She means that they seem to understand very quickly what the activity is all about.

[334] A word that I cannot decipher.

R: [laugh] What's this Ohhhh?

D: I really struggled to find something

R: [the one] with the Braille system, wasn't it?

D: Yeah… Yeah…

R: You don't like it anymore?

D: Well… I don't know, I felt like as I really struggled to pick something that I actually really liked; originally I was looking at 'palindromic dates' but I thought well…

R: Something similar with Mandy's [activity]

D: Yeah… But I didn't; I was looking at it and I thought "okay, it's interesting", **to me** it was interesting but I just thought… there's no… I don't know… it suggested in the book that I found that somebody try and find out which year had; and **I thought well… that's just a hell of a lot work! And I just thought I've only got 20 minutes**[335] and I don't want people to look at it and think "really Diana you want me to do **this?**" [laugh] So, I felt as I was really struggling and then I saw the Braille thing about 5 minutes before the lesson ended.[336] And the teacher said "oh that's quite a good one". And **I'll admit**, I didn't research it as much as I should have done! And I did it and I don't know, it just seemed really obvious because I haven't… I don't know…

R: Really obvious?

D: I don't know I just…

R: What is it? You got a kind of bad feeling about it, is there something wrong with it? I mean…

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[335] My emphasis.

[336] It was a lesson dedicated to providing ideas in order that the students would choose the material that they would use for their presentations.

D: I don't think there was anything wrong with it! Because I suppose… **it lasted 20 minutes as it was supposed to and we all got kind of an answer to it and everybody seemed to understand it**[337] but…

R: Do you mean that it would be more successful if not everybody seemed to understand it, if nobody would understand it or if half of them or just one [would understand it] or what? [laugh]

D: [laugh]

R: When would it be successful [to your opinion]?

D: I don't know…

R: When would you be happy with it?

D: When would I be happy with it?

R: Yeah

D: Well, **because I hadn't put as much thought into it as I should have done,**[338] which I'll admit, I completely admit I should have put a lot more thought into it than I did; and **I never thought to see the pattern,** you know the… if you've got 6 dots then it's only got one possibility, if you've got 5 dots then -I can't even remember how many possibilities you had- and then obviously I saw a pattern in it but… **in a way I was just going with the flow of the class and what they found because I didn't really have any answers to it, I didn't know what was going to happen!**[339] I just saw it and thought "yeah I'll _[340] off with that"

R: Some people came out with a formula there

D: Yeah, so…

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[337] My emphasis; in this statement seems to be included whatever Diana thought as important in her presentation, in the way that it was processed.

[338] My emphasis.

[339] My emphasis; Diana is talking about the formula $\frac{n!}{k!\cdot(n-k)!}$, which calculates the number of different ways that k dots are marked in a total number of n dots (n equals to 6 in the Braille system that Diana presented)

[340] Unclear word.

R: This is the way… they thought… a few people… they found the formula because it's combinatorics there, 2 dots, 3 dots over six, 4 dots over six, 5 dots over six. So it was interesting [for] quite a few people there

D: Yeah…

R: But you're not happy with that [her activity], are you?

D: Well, I don't know what I'm going to say about it when I have to write up my… because we've got to put it in a… final work and I've got no idea what I'm going to say about it

R: That's what worries you about it actually, not the activity itself…

D: Well, **partly**, the activity was alright, I was doing it and it seemed like a good idea; **I felt okay about it before I started doing it and then I started doing it and I just thought "this is rubbish"**[341]

R: Do you usually think like that?

D: **Most of the time**, yeah…

R: Really?

D: Yeah! [laugh]

R: Mmm…

D: [laugh]

R: [laugh] Tell me please how do you think a teacher would be, because in a sense, did you feel like that? That it was…[342]

D: **In a way I liked it that I didn't know the answers because I was learning with the class.**[343] And it was nice because I had certain ideas of… sort of a rough idea of what might happen but then it was interesting to see everybody else's ideas; so in a way **I suppose it was a good thing because I had no preconceived ideas of what might happen, it left me open to all the**

[341] My emphasis.
[342] If she was thinking as if she was acting like a teacher.
[343] My emphasis.

possibilities, so I wasn't saying "right, well this is the answer to it"; it wasn't this definite answer, it was like "right, like okay let's just explore it together and see what happens". So it left me open to suggestions[344]

R: So how would you think yourself as a [teacher]… what is your model of teaching? (43:00) Is it something similar to what you have seen in these presentations or your presentation? A teacher of mathematics, that is.

D: I think when I'm teaching **I need to get over the urge to give answers… not straight away.**[345] If somebody doesn't know the answer to something I'm very… I **want** to be able to tell them rather than help them to understand

R: To understand themselves?

D: Yeah, I sometimes have an urge to say "the answer is this"; rather than try and explain it. I am controlling the urge I am doing quite well with that; I'm having training when I'm teaching [laugh] but I… but I don't know... I don't really know because I have to admit in the NoM sessions I have sometimes been **really frustrated** because I sometimes felt as though… I've asked a question **and he's answered it with a question!** And sometimes it's just really **frustrating** because I think, well, "am I doing it right or am I doing it wrong?" and he says "what do you think?" And I'm thinking that "I don't know, that's why I'm asking you!" And it frustrates me

R: [laugh] Yeah, all he means I suppose is "do whatever you want…

D: Yeah, yeah

R: …it's all about your way of doing it, not me, telling you how to do it"

D: Yeah, I understand that, I do understand that but I had times that I found it really frustrating

R: Because you need somebody to really tell you **the answer?**

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[344] My emphasis.
[345] My emphasis.

D: Yeah, because I feel I've got to a point where I **really** don't understand it anymore and I don't necessarily need an answer! Just a little bit of **guidance**; maybe **guidance** would be a better word rather than answer, you know, **a push in the right direction;** but sometimes I felt as though I wasn't getting that and it frustrated me

R: [laugh] Yes, yes I know what you mean. What about the activity with the moon and the earth and the sun?

D: [very spontaneous laugh] Oh I think me, Ashley and Mandy just… we were in our own little world with that then, weren't we? To us it made perfect sense, what we were doing made perfect sense to us; I mean **I enjoyed it** because I felt as though… I mean obviously I am not claiming to be an expert but I kind of felt as though I knew the basic workings of the solar system and **I was surprised how much I actually learnt from that lesson**. Because everything that I've done in the past in school and everything it'd all been… so like when the sun was moving round I'd never really thought about the seasons and where the sun is positioned and you know, I'd never really thought about that; so it was really interesting seeing that! And actually **having the visual there for it**[346] whereas in the past I think I haven't had the visual for it, it's just been "you've seen it in the textbook"

R: The visual?

D: The visual, yeah, you know how we had the light in the middle, which obviously was the sun and the planets, but when I've done anything in the past with school it's just been in textbooks; so we never actually _[347] anything practical, like we did in that lesson. And then, obviously, as pretending to be the earth and orbiting around the sun was interesting

R: And how did you learn, you think?

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[346] My emphasis.

[347] Unclear word.

D: Well, like I say, to me what me, Mandy and Ashley were doing it made perfect sense; all three of us, it didn't seem to make sense to anybody else but to all three of us we knew exactly what we were doing

R: Did you agree with each other?

D: All three of us agreed, yeah; so **our logic was, because we agreed, it was right**.[348] Because all three of us agreed on it and we all knew what each other meant, then it made sense. And it was okay, it was fine

R: Could you explain it to anybody else, what you were thinking, how you concluded to whatever you concluded?

D: I could try, well, the thing is, when we did what we were doing… and then, after that we went to the… you know, the polyestering balls we had

R: The what?

D: You know, the polyestering balls, and we had all of them, and the light was in the middle, well, that explained what we were doing. So, I think what it was that we knew what was happening and that when that part of it was explained to us with the 'summer-spring-autumn-winter' we knew what was going on and we were saying what we've just said. But nobody else seemed to understand our way of interpreting it. You know, when we were stood up and we were hoping around, well, our explanation of that was exactly the same, more or less exactly the same as the explanation that we were given when we were all standing around looking at these polyestering balls

R: I had the sense that Mandy was moving like this[349] [I tried to literally show her]

D: Yes she was

R: And she was not moving like this[350]

D: No, she wasn't

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[348] My emphasis.

[349] See Mary's drawing during the activity. I will now describe to Diana an episode from the "*Earth-Moon-Sun"* activity in a humorous way.

[350] That is without any axis shift.

R: And I think that at the same time Ashley was thinking that the earth was moving like this… not the axis, as a whole;[351]

D: Alright…

R: So I thought there was a different understanding and I'm checking it now by asking you because you were there

D: Maybe it was just me and Mandy

R: I was not there at the second part of the activity; in fact during the first part of the activity I was talking with Ashley, and you and Mandy were talking to each other

D: Yeah! [laugh]

R: We were two (split) groups and…

D: Yeah!

R: …as soon as I finished[352] I thought "oh, what are we doing here? We are one group, not two…

D: Yeah![353]

R: …subgroups!" So I turned to you and I started talking to you, and you were a bit annoyed actually, I think [laugh]

D: Oh was I? Sorry [laugh]

R: No, it's okay [laugh]

D: [laugh]

R: "Who is this guy? He just entered the conversation"[354]

D: [laugh]

R: I was just trying to include everybody,[355]

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[351] The latter way that was just described.

[352] When I finished talking with Ashley (see *Reflection on earth, moon, sun activity* file)

[353] Diana recalls the episode that Andonis is describing.

[354] Andonis means that they were involved in their own investigation of the activity (there was no conversation actually, Diana and Mary were staring at the "polyestering balls" in silence).

[355] See *Reflection on earth, moon, sun activity* file for a full description of this episode

D: Yeah!

R: you know, I didn't want us[356] to draw our own conclusions

D: Oh sorry, I didn't mean to [laugh]

R: You were like this… sitting and looking at the [laugh] light in the middle and Mandy was at the other side, like this…[357] you were looking

D: I think it was because **I think we were getting a little bit frustrated because to me and Mandy it made perfect sense but we couldn't articulate to other people.[358]** I think we were quite frustrated (50:13)

R: So you think that earth is moving like this?[359] Or like this?[360]

D: I can't remember what conclusion… No, I think we came to the… I think we came to the conclusion… Oh… No, I think it was going like that

R: But Mandy was [thinking that] it was walking like that, I thought, as far as I remember, she was going like this, the whole body was moving and she was using the left [leg]

D: Oh, that's because when Mandy was doing it like that we were assuming that her leg, like going through her body was the axis

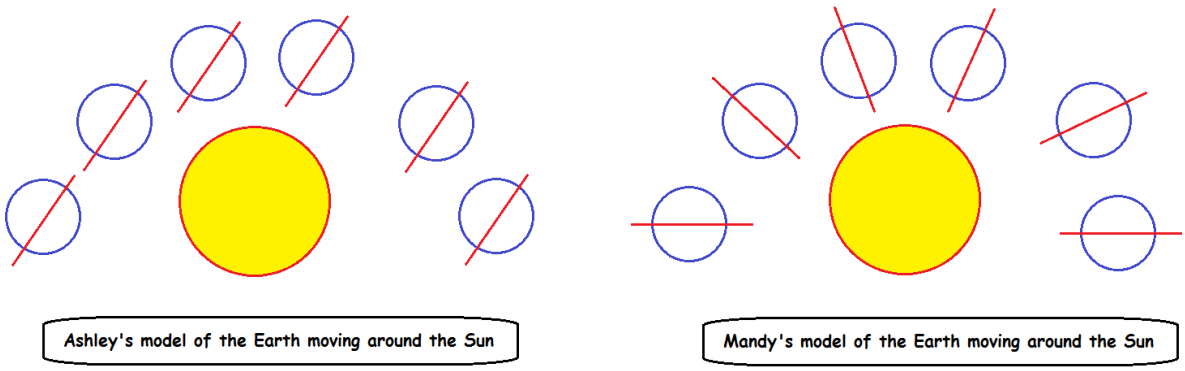

Figure 3 Different models of Eartth moving around the Sun

[356] That is, Andonis and Ashley to draw their conclusions separately from the rest of he group
[357] Andonis is describing how they were sitting when he interrupted them
[358] My emphasis.
[359] That is, the way that Mandy described it.
[360] Without any axis shift.

R: That's right, that's what I thought. And I thought "okay that means that the axis is moving", as the axis was going like this[361]

D: Oh, maybe that's why it made no sense to anybody else then!

R: But that was a way of thinking about it; I'm not saying one [way] is good, one [way] is wrong, I 'don't know anything'; I say just what I saw. And then I thought that Ashley was thinking that it's going like this, I cannot turn my finger,[362] you know

D: [laugh]

R: [laugh] and now you say to me that you all agreed and I think "how could you agree if you had different models"?

D: Well, as far as I understood it **we all did agree; but maybe thinking about it now, maybe I was just agreeing with Mandy and maybe we just assumed that Ashley agreed with us[363]**

R: It might be… But what were you and Mandy agreeing on? That's what I would like to know… on which kind of movement?

D: I think we eventually agreed that it was going like that

R: In the end?

D: Yeah, that's what we eventually agreed on as far as I'm aware. Yeah… I think maybe when we modeled it the first time we actually meant that; we actually

R: And why were you moving your head so that it always looked at the light?

D: I can't remember what that was for, was that the moon or something?

R: It was probably the moon because the moon is always having the same…

D: The way that we… it was all the 3 of us that were doing that, the same, weren't we? The way that we saw that was… What we were doing? So it was the moon that we were looking at weren't it?

R: Yes

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[361] See Mandy's model in the Figure 3.

[362] That the axis is spinning around without shifting (see Ashley's model in the Figure 3).

[363] My emphasis.

D: Right! So we were saying that our faces were… not the moon… we were saying our faces… I think that we said our faces were the side of the Earth when it's… night time or something like that… We got told that a certain side of the Earth or something like that always had to be facing the moon or the sun or whatever it was; so that was the reason that we were going so that we were always facing the object that was hanging of the ceiling

R: You were told that the moon was always facing the earth with the same side; one side of the moon is the dark side, you never see it

D: Yeah… was it? **I don't remember now! It seems such a long time ago**[364]

R: It's okay, I don't want to put you in…

D: No I was… we've been told at some point that basically our faces or whatever always had to… always had to be facing whatever object it was hanging of the ceiling; which is why we were turning like that so we were always facing the… and we were looking the way our bodies were going when we were doing that

R: And why wouldn't you turn your bodies instead of 'breaking your necks'? Was the body a separate part of the planet?

D: No… I **think** the reason we were doing that was because for us to move round in a circle[365] the only way we could do that and keep our faces facing that object was to twist our bodies round. So our bodies were actually not part of what we were trying to… it was just the fact that we wanted our faces to always be facing this object; and because we had to turn in a circle like this, that was why our bodies were moving as it were (55:08)

R: So your head was the moon?

D: I think that was the general gist, yes

R: [And] not the other part of the body; the other part of the body was carrying the planet?

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[364] My emphasis; the interview would probably be more successful concerning the planets activity if it would take place soon after the activity.

[365] Diana describes a circular motion, not an elliptical one.

D: No, no the body was just in the way. We needed our faces to be constantly facing this object, which was why…

R: Yes, yes. So, the final question, if you like

D: Yeah

R: What did you think about these sessions [the NoM sessions]?[366]

D: When we first started doing them I enjoyed the investigation that we were doing but I have to admit as time went on I started getting more and more frustrated. And I think, partly because of the reason I said before; that sometimes I felt as though I was never given any kind of push. And I think another part was because I always felt as though I work so much slower than everybody else. And I know it's not all about speeding, you have to do learn at your own pace but I always felt as though I was that far behind everybody else; and I don't know to me I never seemed to understand anything, it always took me about half an hour longer than everybody else. And, like I say, as soon as…

R: Everybody else! Everybody else? [laugh]

D: The table I was sitting on then. But then, as soon as I understood it I feel as though the lesson was over and that was it! Nothing was carried on with it. But that was obviously my own doing because I've not gone away from the lesson and thought "right, well I'm going to carry on with my understanding of it I've just finished". So it's… *I am obviously to blame because I'm in charge of my own learning.*

R: I think that these sessions were introducing an alternative way in learning and…

D: Yeah

R: And I was just asking how you feel how you think about this alternative way if you like, now that the sessions are over; how you feel about this alternative way of learning and constructing knowledge

[366] It is the last part of the interview and Diana will describe her impressions of the NoM sessions and the effect that they had on her.

D: I think it's good because **encourages you to learn for yourself** and **you can then see how you learn** and apply it to other things; other than just maths you can apply it to different things whereas… I'm thinking **in schools and everything it is as we've spoken about before you are not developing necessarily your understanding, you are been taught to pass an exam!**[367] So, I think it was… **I think part of it I found it frustrating because I never had this kind of teaching before; throughout college and throughout school it's alright "you need to do this to be able to get this grade, you need to be able to do this to be able to get that grade" whereas this was a lot more of my own thought processes and understanding and…**[368] So it's good but **I think because it's new**[369] and maybe because I'm a bit older and I've never come across it before

R: A bit older? So you think the others **have** come across it before?

D: Not necessarily! I don't think Donald or Ivan have ever come across it before, but because I've got through say 9 years more teaching than Donald and Ivan for example, then, you know on different teaching methods I've still never come across it. So, maybe it's harder for me than it is for them because they're more… open to it?

R: What have you discovered from these sessions? Have you discovered anything in your personal way of…

D: I think, **personally**, when I put my mind into it **I can actually come up with things myself rather than relying on other people and saying "well, what have you got for this and what have you got for this"; and I think I've learned to trust myself a little bit more. Because a lot of the time –and it's not just with maths, it's with other subjects- I've been very cautious when I found an answer, if my answer isn't the same as everybody else's I've just assumed mine is wrong! Because it's not the same as somebody else's**

[367] My emphasis.
[368] My emphasis.
[369] My emphasis.

[answer].[370] But I think the lessons that we've had and because of the way that we have been taught to think for ourselves, and it is a lot of independent thinking, that **I think I've possibly become a bit more confident with what I'm doing**; because I'm thinking "well, **because I've got a different answer to somebody else it doesn't necessarily mean that it's wrong, it just means I thought about it in a different way**".[371] So it has, yeah it's been good for me! **I have enjoyed it but it's been a hard work** [laugh]

R: [laugh] (1:00:35)

D: It's taken me a while to fully appreciate it! Let's put it that way

R: Do you think that you will apply any of these things to your teaching?

D: Yeah, well, like I said before about the… about my need to just give people the answers but… **I think I have realised from these lessons that it helps people more if you help with their understanding of it rather than just give them an answer straight away; and naturally make them think about it and what they are doing and why they are doing it, which obviously is very important with wanting to be a teacher**.[372] **It has been good!** Yeah!

R: After so much pain and suffering…

D: After so much pain, yes! It was worth it, although I won't admit that to the teacher [laugh]

R: Thank you very much for your… [cooperation], I've enjoyed it

D: You are welcome, no problem.

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[370] My emphasis.
[371] My emphasis.
[372] My emphasis.

Appendix D. Mary's lengthy interview, 19/5/2011, concerning the curves activity and the stellated octahedron activity.

Paragraph 1. Mary talks about the curves activity, where every point is equidistant from a wall and a fixed point

Researcher: Which one is your favourite one?

Mary: This one. This one is definitely my favourite one

R: The 'garden-path' one. Why?

M: Because it came to a formula

R But some other [activities also] came to a formula

M: My other favourite [activity] was the equidistance between the fixed point

R: Did you sort the problem? [she had a problem with a 'corrupted' computer file]

M: I've done it now, I finished

R: Where is the original?

M: This is the original

R: Oh yes, yes [I recognised it]. You were the most silent person in this activity you were just sitting there and nobody knew what was going on there in your mind; 'was it working at all?', 'what was happening there?'. And then you came out with all this

M: I could not get it; on that day I could not get this. At all!

R: You remember what was happening?

M: Yeah, yeah. Ashley was demonstrating and you were demonstrating and in fact it wasn't happening!

R: Yes, it wasn't happening and I thought "what's wrong?". I was using the Pythagorean Theorem,

M: Yeah

R: Ashley was moving up and down, Yvette came and Mary was just sitting there... and then you came out with this; when you came home what [happened]? It was

a 'revelation'?

M: Well I thought I better try and understand it

R: And then it was all easy?

M: Yes!

R: Suddenly; how did that happen? How do you come from the situation where everything was so complicated and incomprehensible...

M: I think it was because we've done like 3 different things on that day, haven't we? I was still thinking about the 1st one and then a little bit it was like I was trying to understand the 2nd one and this 3rd one came just like a... It can't work!

R: You mean in the same activity we were doing many things

M: Yeah

R: Yes, we firstly did this where you have equidistance from 2 points, and then from a wall, from a line, and then came this and then we moved to another case

M: It was all keep going... too much

R: So you couldn't concentrate in this particular one

M: Too much. Yeah

R: And then? Everything was sorted out?

M: Yeah, when I got home I just drew **this little diagram,**[373] like... obviously the wall... then the cross… and then we knew that this is 10[374]… and obviously this is it[375] and then **I think what I couldn't understand on the day was that this wall went on forever!**[376] Like… do you know what I mean?[377]

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[373] My emphasis; Mary showed me a piece of paper that I reproduced here –see **Figure 1**.

[374] She means the distance between the fixed point and the wall.

[375] The diagram serves as a representation of the situation that took place during the session; an obvious attempt to **visualise** the situation that took place.

[376] My emphasis; it was through the diagram visualisation that she managed to get hold of the abstract mathematical presumptions of the problem, to transcend the limited representation of the classroom and the limitations of the particular situation in the classroom.

[377] By drawing the "little diagram" she realised that the wall was just the representation of a straight line and she was transferred from the **concrete particular case** to the **general abstract case**. In other words her

R: Yeah

M: It went on, it wasn't just a fixed point like this was; so then and obviously these 2, these are the same,[378] 10 [units] away from the fixed point, [which is] on the horizontal, I'm going to make 10 away from the [wall]…

R: This is a circle here

M: Yes

R: Like that, eh? 3 points, one is on the wall and these 2 equidistant from the fixed point, yes?[379]

M: So this point, this point and this point… so then, I was like "there must be another point; so what would happen if I'd change… if I moved this point so [that] it was on the line of y=4", yeah? So then I changed on here [another diagram] **and I knew[380]** this distance to the wall is going to be 6 now.

R: Yes

M: So using Pythagoras

R: Yes

M: this is going to be 4 isn't it?

R: Yes

M: I need this to be 6 and this [has to be] equidistant to the wall

R: Yes

M: So then what's the x coordinate going to have to be for that?

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intuition was limited to the concrete example during the session, making it difficult to think of the problem in abstract mathematical terms. The abstract representation of the diagram allowed her to further her perspective and mathematise the problem.

[378] She means the distance between any point of the curve from the wall and from the vertical axis.

[379] I'm choosing to interrupt her **now that her discourse has found a rhythm**, so that I'll have a recorded description of the diagram that we are engaged in, for later reference; it is apparent from what she says after my interruption that she was not influenced from it. The 3 points that I mean here are the points with coordinates (0,5)—halfway between the fixed point and the wall—, (-10,0) and (10,0) [I'm describing the first points of the diagram that Mary found belonging to the curve that she was looking for –see Figure 1]. The circle that I just mentioned has its centre on the fixed point and its radius equalled 10 units.

[380] Her emphasis.

R: Aha!

M: So then I did that and then I got these 2 points.[381] And then I changed the y distance again to 7… to… I changed it to y equals 3,[382] for these lines; this changed to 7, so the hypotenuse changed to 7, so then I…

R: I'm very happy

M: So then it just came like that

R: I'm very happy because **I can see here all the ideas that we were talking about**—and some of them of course I introduced them and I remember them

M: Yes

R: Like "why don't you find many points and then figure out what kind of curve or line…

M: Yeah

R: …or whatever this is". So that's what you are doing here, you're finding more and more points, and **then** you will find out what is going on

M: Yeah. And once I got up to the line where… on to the x axis, see? I changed it all to it like the graphs so that I could understand it. So then, once it was on the x axis I actually didn't think that you could have a point beyond that[383]

R: Aha?

M: It turned out that it just keeps going

R: Why can't you have a point beyond that?

M: I didn't think you could because I was like "but it's always going to be close to the fixed point!" But it's not because this doesn't go on forever whereas if we had a point beyond the wall then it's always going to be closer to the wall; it's never going to be the same distance between [the wall and the fixed point]

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[381] She means the points $\left(-\sqrt{20}, 4\right), (\sqrt{20}, 4)$ [see her notes for the use of Pythagorean theorem in order to calculate the value for the x coordinate]

[382] y equals 3, therefore 7 is the distance from the wall and –consequently– the fixed point.

[383] She means beyond the x axis.

R: That's right! So if you go beyond this 'border', if you call it like that, the distance from the fixed point will always be smaller than the distance from the wall, is that what you're saying?[384]

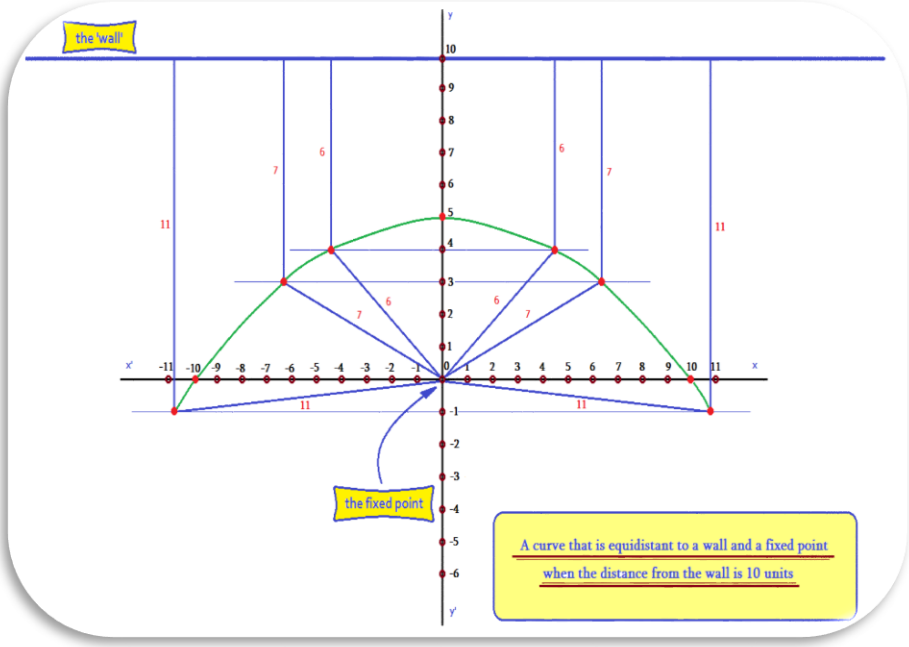

Figure 1: Mary's "little diagram"

R: No! If we went beyond **the wall**

A: Beyond the wall, yes

R: The distance to the wall is always going to be smaller than the distance from the fixed point

R: And… and I'm adding…

M: On the other side?[385]

R: On the other side

M: No!

R: Is this what you're [also claiming]

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[384] I'm trying to understand what she is talking about, thinking that she means the fixed point instead of the wall.

[385] Beyond the x axis, that is the 'level' of the fixed point.

M: No, because I got to this point and then I was like "Oh! I'll try using the same method that I did"

R: Yes

M: Then we went to here, so I said, so if we moved beyond the x axis the distance from the wall is going to be 11, so then the hypotenuse is going to be 11 and this distance here is going to be 1. So then what that has to be?[386] And it worked. So then it carried on to show that the curve is continued (7:11)

R: So it does continue on this side [and] not on the other side. I like it, this is mathematics I love them, you know what I mean

M: Yeah!

R: So what kind of curve is this one?

M: x squared. And then

R: Yeah, oh I'm sorry, just continue and then?

M: This is lovely; and then I literally disrupted my ideas. So it's clearly a curve it's going to be in the form of x^2; however an x^2 graph is like a cup shape

R: Yes

M: Yeah? This is the dots and this is going to be a negative x^2 so I have the equation $-x^2$ but we know on a normal x^2 graph it's going to cross the point at (0,0). This one didn't, it was translated up to 5 so I had a translation of (0,5); so then I got an equation of $y = 5 - x^2$ alright? But then I tried the point (10,0) so, this one here and when I put that to the equation I got -5=-100, which obviously doesn't… So then to get to 5 you have to just divide 100 by 20. So then I came to an equation of $y = 5 - \frac{\mathbf{x^2}}{\mathbf{20}}$; it's the equation of that graph. Yeah?

R: Did you test more points to see if it fits?

M: Yeah, yeah; and then I used $\sqrt{24}$ and then put that in. So I only used $\sqrt{20}$, I knew that the coordinate was 4 from my graph; right? And then it came out that y=4. So it was clear that that was the actual equation

[386] She means the x coordinate.

R: Because you tried one more point and it fitted

M: Yeah, yeah, yeah; and then I did it so that the wall was only 6 away from the fixed point; and then I did it all over again. Before I did this I thought that the general formula, it doesn't matter the distance between the wall and the fixed point at all, if we call that distance a, [then] the formula—any, no matter what the distance is—is going to be $y = \frac{a}{2} - \frac{x^2}{2a}$

R: Wait; you say that very quickly and I'm not English

M: This is the equation that I got from the last graph; so now we're saying this is the wall and this is the fixed point then the distance here is a. So this, in this occasion, a was 10 wasn't it?

R: Yes

M: We've got 5 so it's going to be a over 2.

R: Yes

M: And then the x^2 is the same and then we've got 20 on the bottom so it's going to be 2 lots of a.

R: Which is 2a

M: Yeah, so then I thought the general formula was going to be $y = \frac{a}{2} - \frac{x^2}{2a}$

R: Alright!

M: But that was just without trying any other… so then I tried the… wall… the fixed point being on the 6 away from the wall. So then I got some more equations like this

[Mary explains how she used the same method in order to find the formulas for the cases where the distance from the wall is 6, 4, and how her general formula also worked for these cases]

R: Can you call this a proof?

M: Yeah

R: Yeah?

M: I proved it

R: Alright

M: And then this is just putting… like…

R: There are many nice things in this procedure; I love it really because you have used the Pythagorean Theorem, many points to find the curve, testing and translating everything to a constant which is a, many nice things (14:00)

Paragraph 2. Mary talks about formulas and 'pure' maths

R: So this one [the curve that all its points are equidistant between a wall and a fixed point] is your favourite one?

M: That one and 'garden-path', it's kind of a joint one. I like this one more because it's finished

R: You like it because it's finished?

M: Yeah; but this one [the 'garden-path'] I'm thinking: "What if I do this? What if I did that?" It could go on forever Andonis, I was even doing that last night

R: And why do you like the 'garden-path' one?

M: Because it came to a formula at the end; I like it when it comes to a formula like… the 'New York cop' I didn't like that one

R: Why do you like it when it comes to a formula?

M: Because it's **satisfying**[387]

R: You like it because it's satisfying

M: Yeah!

R: That doesn't make sense!

M: It makes perfect sense! [laugh]

R: And you are satisfied because you like it!

M: Yeah, you feel like you've achieved something; if you don't come to a formula it's just leaves… you keep going

[387] Her emphasis.

R: [You mean] "What's the point" if you don't get a formula. So the formula is important for you?

M: Yeah, kind of

R: In what sense?

M: Because it's like you've got something at the end; like you can say **"this is true!"**. Like for that I am positive that I can say that if this is true and if this part was a, you could stand on anywhere on the line $y = \frac{a}{2} - \frac{x^2}{2a}$ and you would be equidistant from the wall and a fixed point; I can **definitely** say that, I would make **no assumptions whatsoever** about the 'New York cop'; at all!

R: Forget the 'New York cop'; I want to understand how you feel about the formula it's very important

M: That was it! [Mary laughs]

R: Wait a second because I have a teasing question: you can have results without having a formula

M: Yeah you can but…

R: You can say "when you do this you'll have that. When you do that you'll have the other thing". **You have results** but **the formula is a different form of results**

M: Yeah!

R: Which… I mean, I am with you [i.e. I have the same perspective that you have][388]

M: [laugh]

R: I just want your… your… [opinion]

M: I'm trying to think about it and put it in words [silence]

R: I'm not going to speak I want your… your words [silence]. What is it with a formula that makes it 'important', makes it… special [silence]

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[388] She is thinking about it and I'm trying to encourage her.

M: I don't know. **I do know** but

R: I mean [you may] give an example to explain I mean… I mean like "I like this because aha! There, you see? The formula is there and so I can…" [silence]

M! I just gave you that example!

R: Can you enrich the example?

M: Okay! Pythagoras, he states that a^2 equals b^2+c^2; so **we know** that for every right angle triangle a is going to be bigger than b and c; for sure! **Definitely!**

R: Yes! Yes! And you can know even more than that

M: Yeah! **It can take you to so many more things to think about**;[389] it can take you to so many more things just like

R: So is it the control you have

M: **Yeah!**

R: over the… phenomenon that the formula gives you that fascinates you about it

M: Yeah

R: Are these my words?

M: No, I know what you mean and it's also why… **it's just the fact that you feel like you got something in the end [silence] like you've come to something.**[390] I can't explain it to you

R: You mean that something is as vague as 'strange' before, as soon as you get the formula you've pinned it down

M: Yeah, you narrow it down to just a small—maybe not simple—but just the size is

R: Which does what to the phenomenon?

M: **It brings it to a more manageable size**; in the teacher's words [laugh]

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389 My emphasis.

390 My emphasis.

R: Your words

M: But it comes from the teacher.

R: But it comes from the teacher, right? It makes it more manageable and it makes it…

M: **Approachable!**

R: Approachable; and it brings results out of it

M: Yeah

R: Yes, while before it is a 'chaos'

M: Yes! **It could be anything!**[391]

R: Without the use of many words like "if you do this and you do that"; you just have this elegant

M: Yes!

R: Little

M: Just a couple of xes[392] a couple of ys[393] a couple of vehicles

R: So you like the xes and the ys and

M: Yeah

R: All the variables and all this

M: Yeah

R: Formal, 'real' mathematics

M: Yeah

R: The algebra also is probably; is it the algebra fascinating for you?

M: Yeah

R: What is your favourite field in mathematics?

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[391] My emphasis.

[392] Multiple tense for 'x'

[393] Multiple tense for 'y'

M: What do you mean?

R: I mean there's algebra there's analysis there's geometry, there's calculus, there's statistics, there's… you name it

M: Right, well, **I don't like applied mathematics**[394]

R: You don't like applied mathematics

M: I don't like it; I don't like mechanics and decision. I can do it but I don't like it

R: So you prefer **pure** mathematics

M: Yeah, **pure**. And I'd say… where would imaginary numbers could that be, in algebra would it?

R: Complex numbers is a realm on its own I think

M: I like that

R: You can have complex analysis, complex algebra; so you like imaginary numbers too

M: I like just **pure** mathematics; and it's not that I struggle with the applied stuff it's just that it doesn't interest me, that's all (27:20)

Paragraph 3. Mary talks about the construction of the stellated octahedron

(49:20)M: I like this one

R: Which one is it?

M: When we had to build those shapes

R: "To build those shapes""?

M: You remember Andonis, the stellated tetrahedron

R: Oh, to build the shapes, to construct them!

M: Not them ones that were hung in there; the other one when we were in that

[394] My emphasis.

horrible room, in that building 0.5 room and the teacher gave us the…

R: the string

M: No, we had to draw the circle with the just our hands like a compass

R: Oh

M: And then he gave us those instructions to build the stellated tetrahedron

R: Yes, yes, yes that very small room at the bottom of the building there. Yeah! Why? Why did you like it?

M: **Because I found shapes that I never seen before**[395]

R: Aha! So **you have discovered**[396] something there

M: Yes

R: Is that it?

M: Yeah

R: How did that happen?

M: Just like **magic! They were just there!** (50:21) Like… me and Barbara, do you know when we had the pyramid at the bottom and then the shape was turning when you put 2 of them together it made that crazy shape **like a star**

R: Yeah

M: We did it with like…

R: Stellated tetrahedron

M: Yeah, that's it! Stellated…

R: Oct…

M: Octahedron, yeah. We'd put the… we'd got out like little triangles because obviously it was triangular thingy and like placed it in time in sellotape

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[395] My emphasis.

[396] I suspect the appearance of mathematical objects.

R: Triangular "thingy"? "Thingy" you mean thing?[397]

M: Thingy! No I meant triangular shapes

R: Yeah, yeah, yeah

M: Because the bottom was like 4 square based pyramids weren't it? Yeah, so we got out triangular shapes and then placed them inside the thing and then stacked them together and then took it out… but made like a net at the same time; which looks like… if I can remember it looked like this [she makes a draft drawing of a two dimensional shape] Do you know what I mean? And **that was interesting, to see the net when everybody else was just… so, seeing how it changed from that, the net that we had into the stellated octahedron there everybody else had…**[398]

R: When was the moment that you felt that you had discovered something? [silence] When was it, was it the final moment that you felt like something new was there or were you suspicious of something already or…

M: No, it was right at the very end

R: Right at the very end

M: Yes

R: And what do you thing about these shapes; they were interesting or what?

M: Yeah they were interesting

R: Because you never thought of them before?

M: No! I had never seen a stellated octahedron before in my life; apart from a star in the sky but it's different isn't it?

R: Quite! It's not very close actually; you have to have a lot of imagination to think of it as a stellated shape (52:27)

[the interview is interrupted by Diana and Ivan]

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[397] I had never heard of this expression before and I wanted to make sure that I understood it correctly.

[398] My emphasis.

(54:10) A: So the 3rd one[399] was the new shapes, when they came out they surprised you

M: Yeah!

R: You [actually] constructed them though, yeah?

M: Well, we didn't have enough time because me and Barbara were making the net and then we were going to put it into the shape and then we had the net and then we would see how the thing, the stellated octahedron, and then it was "Oh! This makes this!"

R: Oh so… alright! So you had the… you didn't manage to get to the end of it but it was interesting because you could see where you started and you could see where you'd end up to

M: Yeah!

R: So you had the two stages of the activity simultaneously there to compare them

M: Yeah

R: And that made it fascinating. Is it what I am saying or is it what you are saying?

M: **That is what I said!** (55:18)

[399] The 3rd activity that stood out according to Mary.